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Cohomology of certain moduli spaces of vector bundles

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Abstract. Let X be a smooth irreducible projective curve of genus g over the field of complex numbers. Let M_0 be the moduli space of semi-stable vector bundles on X of rank two and trivial determinant. A canonical desingularization N_0 of M_0 has been constructed by Seshadri [17]. In this paper we compute the third and fourth cohomology groups of N_0. In particular we give a different proof of the theorem due to Nitsure [12], that the third cohomology group of N_0 is torsion-free.

Keywords. Stable bundles; semi-stable bundles; parabolic bundles; conic bundles; Gysin map; Hecke correspondence; Brauer group.

1. Introduction

Let X be a smooth irreducible projective curve of genus g over the field of complex numbers. Non-singular models of the moduli space of semi-stable vector bundles on X of rank two and degree zero have been constructed by Narasimhan–Ramanan [7] and Seshadri [17]. In this paper, we propose to compute some of the Betti numbers of the non-singular model due to Seshadri. In particular we prove the following theorems.

Theorem (A). *The third cohomology group of the non-singular model N_0 of* [17] *is torsion-free, $g \geqslant 2$.*

Theorem (B). *Let B_1 denote the Betti numbers of N_0. Then we have:*

$$B_3 = 2g, \quad B_4 = \binom{2g}{2} + 4, \quad g \geqslant 4.$$

Theorem (A) is due to Nitsure [11]. He proved this for the non-singular model of [7]. By Artin–Mumford [1], the torsion subgroup of the third cohomology group of a smooth projective variety is a birational invariant. Therefore any non-singular model has torsion-free third cohomology.

We present here a considerably simpler proof of Theorem (A) using the model of [17]; in fact, this was the initial motivation for this work. However we should point out that the general line of attack is as in Nitsure [11]. An extension of the ideas involved in the proof also yields Theorem (B). For computing B_4 and B_5 we make use of the results of Kirwan [5].

Nitsure showed independently that $B_3 = 2g$ for the model of [7] (cf [12]).

In Appendix 1 we present a proof of Theorem (A) due to Coliot-Thélène which is independent of the non-singular model chosen.

Theorems (A) and (B) are of interest in understanding the rationality of these non-singular models of the moduli space of vector bundles.

The layout of the paper is as follows. Section 2 of this paper gives various properties of the non-singular model constructed in [17]. In §3 we construct a canonical generalized conic bundle on the non-singular model N_0. In §4 by using a result of [9], we prove Theorem (A) and show how to compute the Betti numbers of the open subset Z of N_0 lying over the stable bundles and the bundles in the non-nodal part of the Kummer variety. In this section, we also give a description of the Hecke correspondence in terms of parabolic bundles as mentioned in (*). This facilitates the computation of the Betti numbers. In §5 we compute explicitly the codimension of the complement of Z in N_0 and thereby compute its Betti numbers.

The author is grateful to Prof C S Seshadri for suggesting this approach and for many fruitful discussions. He thanks A J Parameshwaran for many an interesting discussion. He also thanks Prof. J Coliot Thélène for communicating his proof.

2. Preliminaries

In this section we shall recall very briefly the definitions and terminologies of [17]. The proofs of most of the statements made in this section can be found in [17] or [18]. We state at the outset that for us the ground field of all our varieties is the field of complex numbers.

(i) X is a smooth irreducible projective curve of genus $g \geqslant 3$.

(ii) Let V be a vector bundle on X. A *parabolic structure* at a point $P \in X$ gives

(a) a *quasi-parabolic structure* i.e. a flag $V_P = F^1 V_P \supsetneq F^2 V_P \supsetneq \cdots \supsetneq F^r V_P$.

(b) weights $\alpha_1, \ldots, \alpha_r$ attached to $F^1 V_P, \ldots, F^r V_P$ such that $0 \leqslant \alpha_1 < \alpha_2 < \cdots \alpha_r < 1$.

Call $k_1 = \dim F^1 V_P - \dim F^2 V_P, \ldots, k_r = \dim F^r V_P$

the multiplicities of $\alpha_1, \alpha_2, \ldots, \alpha_r$.

The *parabolic degree* of V is defined by

$$\text{par deg } V = \deg V + \sum_i k_i \alpha_i$$

and write par $\mu(V) = \text{par deg } V / \text{rk } V$.

If W is a subbundle of V, it acquires, in an obvious way, a quasi-parabolic structure. To make it a *parabolic subbundle*, we attach weights as follows:

Given $i_0, F^{i_0} W \subset F^j V$ for some; let j_0 be such that $F^{i_0} W \subset F^{j_0} V$ and $F^{i_0} W \not\subset F^{j_0+1} V$; then the weight of $F^{j_0} V = F^{i_0} W$. Define V to be parabolic stable (resp. semistable) if for every proper subbundle W of V, one has par $\mu(W) <$ par $\mu(V)$ (resp. $\leqslant$).

(*) Mehta V and Seshadri C S *Math. Ann.* **248** (1980) 205–239.

If V_n be the category of semistable vector bundles on X of rank n and degree 0, then we denote by PV_n the category of parabolic semistable vector bundles at a fixed point $P \in X$ and fixed parabolic structure. Recall that, one can choose the weights (α) small enough so as to have the condition 'parabolic semistable' equivalent to 'parabolic stable!

(iii) N is the set of isomorphism classes of $(V, \Delta) \in PV_4$ (Δ a parabolic structure), such that End V is a 'specialization' of $\mathscr{M}_2$—the 2×2 matrix algebra.

In fact, if (V_1, Δ_1) and (V_2, Δ_2) belong to N, they represent the same element of N (i.e. isomorphic in PV_4) if and only if the underlying bundles V_1 and V_2 are isomorphic (cf [17]). *Hence we often simply write* $V \in N$.

(iv) $\mathscr{A}$ is the variety of all algebra structures on a fixed 4-dimensional vector space which are specializations of $\mathscr{M}_2$ and admit a fixed identity element. We have a *canonical group of automorphisms* acting on $\mathscr{A}$, namely the subgroup of $GL(4)$, fixing this identity element.

(v) M denotes the normal projective variety of equivalence classes of semistable vector bundles of rank 2 and degree 0 under the equivalence relation $V \sim V'$ if and only if gr $V =$ gr V'.

(vi) M^s will be the open subset of M consisting of the stable bundles.

It is known that $M - M^s$ is precisely the singular locus of M (cf [6]). The main theorem of [17] is stated below.

Theorem 1. (*Seshadri*) *There is a natural structure of a smooth projective variety on* N *and there exists a canonical morphism* $p: N \to M$, *which is an isomorphism over* M^s. *More precisely, if* $V \in N$, *then* gr $V = D \oplus D$, *with* rk $D = 2$, D *is a direct sum of stable line bundles of degree 0 and the morphism* $p: N \to M$ *is given by* $V \mapsto D$. *Further* $V \in p^{-1}(M^s)$ *if and only if* End $V \simeq \mathscr{M}_2$ *or equivalently (which is easily seen)* $V = W \oplus W$, *where* W *is stable.*

In the course of proving the smoothness of N, Seshadri defined a morphism from a neighbourhood U of a given point of N into A which we shall denote by

$$\varphi^U : U \to A.$$

We shall briefly indicate the construction of φ^U: The functor defining the moduli space N being representable, we have a defining vector bundle E on $X \times N$ of rank 4. Let $f: X \times N \to N$ be the canonical projection and End E the vector bundle associated to the shear of endomorphisms of E. Set

$$B = f_*(\text{End } E).$$

B is the canonical family of specializations of $\mathscr{M}_2$, parametrized by N (see Prop. 5 [17] for details). Consider any given point $u \in N$; then choosing a neighbourhood U of u, which trivialises B, we get a natural morphism

$$\varphi^U : U \to A \quad \text{by} \quad V \mapsto \text{End } V, \quad V \in U.$$

This morphism exists by the so-called versal property of A. Further, let $A_0 = \text{End } V_u$, V_u the vector bundle corresponding to the point $u \in U$, i.e. $A_0 = \varphi^U(u)$. Then, if A_u is the

mini-versal deformation space of A_0, the morphism

$$\varphi_1^U : U \to A_u$$

induced by the versality of A_u from φ^U is in fact *smooth.*

Note 1. By an abuse of notation, in the course of this work, we shall suppress U and the mini-versal deformation space corresponding to each point, and simply denote by $\varphi : N \to A$ the *smooth local morphism* defined above. In fact, we will be using it only in this form in this work.

Note further that these φ^U are uniquely determined modulo automorphism coming from the canonical group of automorphisms acting on A.

PROPOSITION 1

The restriction of the local morphism φ to the subvariety N_0 remains smooth.

Proof. Let J denote the Jacobian variety of line bundles of degree zero on X. Then we have a natural morphism

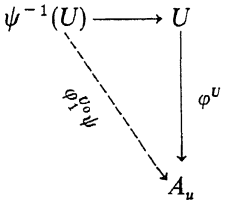
$$\psi : N_0 \times J \to N$$
$$(E, L) \mapsto E \otimes L$$

(that this map is a morphism follows from the universal property of N and the fact that $E \otimes L$ gives a family on X parametrized by $N_0 \times J$).

We claim that ψ is *smooth.* In fact, ψ is *étale.* For, let $\Gamma \subset J$ be the finite subgroup of J consisting of the elements of order 2. Then there is a natural diagonal action of Γ on $N_0 \times J$ which is obviously fixed point free. It is not difficult to see that N is actually the quotient of $N_0 \times J$ by Γ and $\psi : N_0 \times J \to N$ the quotient morphism (note that our ground field is $\mathbb{C}$ and if A and B are smooth complex manifolds and G a finite group acting on A such that B is the set theoretic quotient of A by G, then B is A/G).

This Γ-action being fixed point free, ψ is *étale.*

For $b \in N_0 \times J$, choosing a neighbourhood U of $\psi(b) = u$ in N, we get the following diagram



where A_u is the mini-versal deformation space of the algebra $A_0 = \varphi^U(u)$ in A.

Since φ_1^U, ψ are smooth, so is $\varphi_1^U \circ \psi$. In other words the *local morphism* (again by abuse of notation)

$$\varphi \circ \psi : N_0 \times J \to A$$

is smooth. If $L \in J$, then $\mathrm{End}(E \otimes J) = \mathrm{End}\, E$ and hence $\varphi \circ \psi$ clearly factors through N_0 to give the smoothness of the restriction of φ from $N_0 \to A$. Q.E.D.

Remark 1. Because of Prop. 1, by the same arguments as in [17], we see that N_0 is a smooth-projective variety. We then get an obvious generalization of Theorem 1 namely that $p: N_0 \to M_0$ which is a desingularization of M_0^s, and that it is an isomorphism over M_0^s etc.

3. Conic bundles

DEFINITION 1

Let S be a variety. A *generalized conic bundle* $\mathscr{C}$ on S gives

(a) a vector bundle V on S of rank 3 and
(b) a closed subscheme $\mathscr{C}$ of $\mathbb{P}(V)$ over S, such that, given $s \in S$, there exists a neighbourhood U of s, where $\mathscr{C} \cap p^{-1}(U)$ is defined by $q = 0$, $q \in \Gamma(p^{-1}(U), H^2)$, H being the tautological line bundle for $\mathbb{P}(V) \xrightarrow{p} S$; i.e. $p_*(H) \simeq V^*$ and therefore $p_*(H^2) = S^2 V^*$, etc.

By definition, $\mathscr{C}$ is an effective Cartier divisor and is therefore defined by a section of a line bundle θ on $\mathbb{P}(V)$. Now locally over S, θ and H^2 coincide and therefore by the "see-saw" theorem (cf. Mumford's *Abelian varieties*) there exists a line bundle L on S such that $\theta = H^2 \otimes p^*(L)$. Since $p_*(\theta) = p_*(H^2) \otimes L = S^2(V^*) \otimes L$, *the condition* (b) *above is equivalent* to an element q of $\Gamma(S^2(V^*) \otimes L)$ or that is to say a *quadratic form*

$$q: V \to L.$$

The *discriminant* Δ of q can be defined as a section of $L^3 \otimes (\Lambda^3(V^*))^2$ and locally as the usual discriminant of a quadratic form. The equation $\Delta = 0$ gives locally the *degeneracy locus* of $\mathscr{C}$.

We now introduce *subschemes* on S, namely for $i = 1, 2, 3$, set

$$S_i = \{s \in S \mid q \text{ restricted to } V_s, \text{ the fibre at } s, \text{ has rank} \leqslant 3 - i\}.$$

Then $S_3 \subset S_2 \subset S_1 \subset S = S_0$. If $g: \mathscr{C} \to S$ be the projection, let $\mathscr{C}_i = g^{-1}(S_i)$, $i = 1, 2, 3$. Then we have S_1 to be the degeneracy locus of $\mathscr{C}$, i.e. given by $\Delta = 0$, and $S_2 \subset S_1$ is the singular locus of S_1. The space $\mathscr{C}$ can be described as follows: $\mathscr{C} - \mathscr{C}_1$ consists of non-degenerate conics; $\mathscr{C}_1 - \mathscr{C}_2$ of pairs of lines intersecting transversally; $\mathscr{C}_2 - \mathscr{C}_3$ of repeated lines and $\mathscr{C}_3$ of the whole plane. We call S_i the *canonical subschemes associated to the degenerate loci of the conic bundle* $\mathscr{C}$ *on* S. Accordingly we make the following.

DEFINITION 2

A *generalized conic bundle* $\mathscr{C}$ is of type I if $\mathscr{C}_1 = \phi$; of type II if $\mathscr{C}_2 = \phi$ and of type III if $\mathscr{C}_3 = \phi$.

DEFINITION 3 (cf p. 164 [17])

Let T be an algebraic scheme and $\{G_t\}_{t \in T}$ a family of algebras parametrized by T and defined by a locally free $\mathcal{O}_T$-module G of rank 4. We say that this is a *family of specializations* of $\mathscr{M}_2$ if, given $t \in T$, there is a neighbourhood T_1 of t and a morphism

$f: T_1 \to \mathscr{A}$, such that $\{G_t\}_{t \in T_1}$ is the base change of $\{A_Y\}_{Y \in \mathscr{A}}$ by f, where A_Y is the algebra structure corresponding to $y \in \mathscr{A}$.

Remark 2. By Remark 3 [17], the above definition has an equivalent formulation as follows: Let $T = \operatorname{Spec} R$, and G be an R-algebra with identity e_0 such that the underlying R-module is free of rank 4. Let $J = G/\mathrm{Re}_0$. Consider the canonical Lie algebra structure on J induced by the associative algebra structure on G. This gives a canonical skew-symmetric bilinear map $J \times J \to J$ or equivalently (in our case) an element of $J \otimes J$. Then we say the algebra G gives a *family of specializations* of $\mathscr{M}_2$ parametrized by T, if this Lie algebra structure is defined by a *symmetric element of* $J \otimes J$. Further, the algebra G is isomorphic to C_q^+, q being the corresponding quadratic form. This definition generalizes, in an obvious way, when T is any scheme, and G a vector bundle of rank 4 on T; however, the quadratic form q on J takes values in a line bundle on T.

Note 2. We shall use this reformulation in the course of this work.

Remark 3

(i) Restrict the canonical family B of specialization of $\mathscr{M}_2$ parametrized by N to the subvariety N_0. *Call this family* B_0.

(ii) For $y \in \mathscr{A}$, let A_y be the corresponding algebra structure; then $\{A_y\}_{y \in \mathscr{A}}$ gives an obvious family of specializations of $\mathscr{M}_2$.

(iii) Let $T = \operatorname{Spec} R$ and G an R-algebra giving a family of specializations of $\mathscr{M}_2$. Then by Remark 2, we get a symmetric element of $J \otimes J = G/\mathrm{Re}_0$. This symmetric element naturally gives rise to a symmetric bilinear form on J^* (the R-dual of J) and therefore a quadratic form on J^*. Now J^* being a projective R-module of rank 3, it defines a vector bundle of rank 3 on T. More generally, if we are given an algebraic scheme T, a family $\{G_t\}_{t \in T}$ of specializations of $\mathscr{M}_2$, then we have a canonical vector bundle. V of rank 3 on T together with a $\mathscr{O}_T$-valued quadratic form $q: V \to \mathscr{O}_T$, and thus a conic bundle on T.

(iv) The families B_0 on N_0 and $\{A_y\}_{y \in \mathscr{A}}$ on $\mathscr{A}$ give generalized conic bundles on N_0 and $\mathscr{A}$ respectively.

Notation 1. Denote these conic bundles by P on N_0 and Q on $\mathscr{A}$.

PROPOSITION 2

The conic bundle P on N_0 is locally the base change of Q on $\mathscr{A}$ by the local morphism $\varphi: N_0 \to \mathscr{A}$ of §2.

Proof. This is an immediate consequence of the definitions of φ, B_0 and $\{A_y\}_{y \in \mathscr{A}}$.

Remark 4. Following §3, we introduce the *canonical subschemes*

$$\mathscr{A}_3 \subset \mathscr{A}_2 \subset \mathscr{A}_1 \subset \mathscr{A} \quad \text{and}$$
$$N_3 \subset N_2 \subset N_1 \subset N_0$$

associated to the degeneracy locus of Q and P respectively. Then, by Prop. 2. φ

locally maps $N_0 - N_2$ into $\mathscr{A} - \mathscr{A}_2$ in such a way that $N_1 - N_2 \to \mathscr{A}_1 - \mathscr{A}_2$, $N_0 - N_1 \to \mathscr{A} - \mathscr{A}_1$.

Remark 5. By Theorem 1 [17] we know that $\mathscr{A} \simeq \Phi \times \Lambda$, where Λ is the 3-dimensional affine space and Φ the 6-dimensional affine space whose points are identified with the set of quadratic forms on a fixed 3-dimensional vector space (or algebras of the form C_q^+—the even degree elements of the Clifford algebra associated to the quadratic form q). Therefore we have for $i = 1, 2, 3$

$$\mathscr{A}_i = \{q \in \Phi | \text{rank } q \leqslant 3 - i\} \times \mathbb{A}^3.$$

Note that

$$\mathscr{A}_0 - \mathscr{A}_1 = \{q | q \in \Phi, C_q^+ \simeq \mathscr{M}_2\} \times \mathbb{A}^3 \quad \text{or equivalently}$$

$$\mathscr{A}_0 - \mathscr{A}_1 = \{y | A_y \simeq \mathscr{M}_2\}.$$

Notation 2. We denote the subsets $N_0 - N_2$ and $N_1 - N_2$ of N_0 by Z and Y respectively.

Let $K = M_0 - M_0^s$, be the singular locus of M_0. The bundles here are of the form $L \oplus L^{-1}$, where L is a line bundle of degree 0. Let K_0 be the 'nodes' of K (i.e. consisting of bundles of the type $L \oplus L$ with L^2 trivial). Then

$$K - K_0 = L \oplus L^{-1}, \quad L \in J - \Gamma,$$

J and Γ as in §2. It may be noted that K *is a Kummer variety* of dim g (cf [6])

PROPOSITION 3

The subsets Z and Y of N_0 are precisely $N_0 - p^{-1}(K_0)$ and $p^{-1}(K - K_0)$ respectively, where $p: N_0 \to M_0$ is the desingularization morphism. In particular, $Z - Y = p^{-1}(M_0^s)$.

Proof. By Remark 3, it is enough to show that the subsets $p^{-1}(M_0^s)$ and $p^{-1}(K - K_0)$ of N_0 are mapped locally by φ into the subsets $\mathscr{A}_0 - \mathscr{A}_1$ and $\mathscr{A}_1 - \mathscr{A}_2$ of $\mathscr{A}_1$ respectively. We know that $V \in p^{-1}(M_0^s)$ if and only if End $V \simeq \mathscr{M}_2$, which shows $p^{-1}(M_0^s)$ maps to $\mathscr{A}_0 - \mathscr{A}_1$.

Therefore it is enough to show that, for $E \in p^{-1}(K - K_0)$, End E has the same defining relations as that of the algebra C_q^+, for a quadratic form q of rank 2 on a 3-dimensional vector space.

By definition of the desingularization, the endomorphism algebras of any two points in a fibre $p^{-1}(L \oplus L^{-1})$ are isomorphic. So we consider a point E in $p^{-1}(L \oplus L^{-1})$ where $E = V \oplus W$, $V \in \text{Ext}(L, L^{-1})$ $W \in \text{Ext}(L^{-1}, L)$, $L \in J - \Gamma$. i.e.

$$\begin{aligned} &0 \to L \to V \to L^{-1} \to 0, \\ &0 \to L^{-1} \to W \to L \to 0. \end{aligned} \tag{1}$$

It is clear that points of this type are actually in $p^{-1}(K - K_0)$. Using (1), it is easy to see that End $(V \oplus W)$ has four generators, which in terms of block matrices can

be described as

$$x=\begin{pmatrix}0 & 0\\ \gamma_2 & 0\end{pmatrix} \quad w=\begin{pmatrix}0 & \gamma_1\\ 0 & 0\end{pmatrix} \quad u=\begin{pmatrix}I & 0\\ 0 & 0\end{pmatrix} \quad v=\begin{pmatrix}0 & 0\\ 0 & I\end{pmatrix},$$

where $I=2\times 2$ identity matrix, and γ_1,γ_2 coming from identification of the line bundles in the exact sequence (1). The defining relations can be given as

$$\begin{aligned} &u^2=u, \quad v^2=v, \quad uv=0, \quad u+v=I,\\ &w^2=x^2=wx=0, \quad uw=w, \quad wu=0,\\ &ux=0, \quad xu=x, \quad vw=0, \quad wv=w\\ &vx=x, \quad xv=0. \end{aligned} \tag{2}$$

If q is a quadratic form of rank 2 on a 3-dimensional vector space over an algebraically closed field k then it is easily seen that C_q^+ = the even degree elements of the Clifford algebra of q is a 4-dimensional k-algebra with

$$\begin{aligned} &C_q^+=k+k\alpha+k\beta+k\gamma \quad \text{such that}\\ &\alpha^2=-1, \quad \alpha\beta=-\gamma, \quad \alpha\gamma=\beta\\ &\beta\alpha=\gamma, \quad \gamma\alpha=-\beta. \end{aligned}$$

Now put $a=\frac{1}{2}(1+i\alpha)$, $b=\frac{1}{2}(1-i\alpha)$, $c=i\beta+\gamma$, $d=i\beta-\gamma$, where $i=\sqrt{-1}\in k$. Then a,b,c,d are new generators of C_q^+ with the following defining relations

$$\begin{aligned} &a^2=a, \quad b^2=b, \quad ab=0, \quad a+b=1,\\ &c^2=d^2=cd=0, \quad ac=c, \quad ca=0,\\ &ad=0, \quad da=d, \quad bc=0, \quad cb=c,\\ &bd=d, \quad db=0. \end{aligned} \tag{3}$$

A glance at (2) and (3) proves our claim.

Q.E.D.

COROLLARY 1

$Y\xrightarrow{p}K-K_0$ is a $\mathbb{P}^{g-2}\times\mathbb{P}^{g-2}$ fibration associated to a vector bundle on $K-K_0$.

Proof. Indeed; we claim that, if $E\in Y=p^{-1}(K-K_0)$ then $E=V\oplus W$, for some $V\in\mathbb{P}(\mathrm{Ext}(L,L^{-1}))$, $W\in\mathbb{P}(\mathrm{Ext}(L^{-1},L))$ $L\in J-\Gamma$.

Let $E\in p^{-1}(K-K_0)$; then, End W has four generators x,w,u,v with defining relations (2) as in Prop. 3. Consider $u\in\mathrm{End}\,E$, and let $V=\ker u$. Then V is a subbundle of E and we have an exact sequence

$$0\to V\to E\to W\to 0.$$

It is clear then that W is in fact $\ker v$, $v\in\mathrm{End}\,E$ and therefore we get a splitting of the exact sequence, implying $E=V\oplus W$.

Now using Prop. 1 of [17], V and W cannot be of the type $L \oplus L$ or $L^{-1} \oplus L^{-1}$. For the same reason, since $E \in PV_4$, we rule out $V = L \oplus L^{-1}$, $W = L^{-1} \oplus L$. Hence we are left with $V \in \mathbb{P}(\mathrm{Ext}(L, L^{-1}))$, $W \in \mathbb{P}(\mathrm{Ext}(L^{-1}, L))$ or vice versa.

Note that for $L \in K - K_0$, $\mathrm{Ext}(L, L^{-1}) = H^1(X, L^{-2})$ has dimension $g-1$ and therefore Y is a $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$ fibration over $K - K_0$. The vector bundle to which this is associated has fibre at any $L \in K - K_0$ to be $\mathrm{Ext}(L, L^{-1}) \oplus \mathrm{Ext}(L^{-1}, L)$.

COROLLARY 2

The fibration $Y \xrightarrow{p} K - K_0$ is locally trivial in the Zariski topology.

Proof. This follows from Cor. 1 and Serre (cf [15]).

PROPOSITION 4

Let $P - P_2$ be the restriction of the conic bundle P over points of $N_0 - N_2$ (i.e. Z). Then the total space of $P - P_2$ is smooth.

Proof. By Prop. 2, $P - P_2$ is locally the base change of $Q - Q_2$ (the restriction of Q over points of $\mathscr{A} - \mathscr{A}_2$). Since $\varphi: N_0 \to \mathscr{A}$ is a smooth local morphism, the total space of $P - P_2$ is smooth if and only if the total space of $Q - Q_2$ is so.

Consider any point $(a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{A}^6$. This defines a quadratic form

$$q = a_1 X^2 + a_2 XY + a_3 Y^2 + a_4 XZ + a_5 YZ + a_6 Z^2.$$

We therefore have a conic bundle C over $\mathbb{A}^6$ by considering the conics defined by the quadratic forms. By Remark 4 it is clear that the conic bundle Q on $\mathscr{A}$ is 'essentially' the conic bundle C. Thus we would have proved our claim if we show that the total space of $C \to \mathbb{A}^6 - S^1$ is smooth, where S is the degeneracy locus of C and $S' \subset S$ its singular locus. We have in fact more.

Lemma 1. Let $\theta: C \to \mathbb{A}^6$ be the canonical morphism. Then $\theta^{-1}(\mathbb{A}^6 - (0))$ is smooth.

Proof. Let $P \in C$ be any point. Then P can be given by $(a_1, a_2, a_3, a_4, a_5, a_6, X, Y, Z)$ where not all $a_i = 0$ and not all $X, Y, Z = 0$, P lying on the conic defined by $q = a_1 X^2 + a_2 XY + a_3 Y^2 + a_4 XZ + a_5 YZ + a_6 Z^2$. Taking partial derivatives of q with respect to a_i, $i = 1, \ldots, 6$, we have

$$\partial q / \partial a_i = 0, \quad i = 1, \ldots, 6 \Rightarrow X = Y = Z = 0.$$

Q.E.D.

4. Cohomology computations

4.1 The Gysin map

Let W be a conic bundle of type I (cf Def. 2) on a variety S. This gives rise to a topological Brauer class b_W in $H^3(S, \mathbb{Z})_{\mathrm{tors}}$ (i.e. the torsion subgroup of $H^3(S, \mathbb{Z})$).

Let W be a conic bundle of type II (cf Def. 2). Then if W degenerates to a pair of lines over an irreducible divisor $S_1 \subset S$, the restriction W_1 of W over S, gives rise in a natural way to a double cover of S_1 (cf Lemma on p. 29 of [8]) and $W - W_1$ is a conic bundle of type I over $S - S_1$. We shall denote by 'α' the element in $H^2(S_1, \mathbb{Z})$ coming from this double cover. Consider the part of the Gysin sequence for $S_1 \subset S$ which involves $H^3(S, \mathbb{Z})$, i.e.

$$H^1(S_1, \mathbb{Z}) \to H^3(S, \mathbb{Z}) \to H^3(S - S_1, \mathbb{Z}) \xrightarrow{g} H^2(S_1, \mathbb{Z}).$$

Then we have here the

Theorem 2. (*Nitsure* [9], [11]) *Let W be a conic bundle of type II on S. If the total space of W is smooth, then the image of $b_{W-W_1} \in H^3(S - S_1, \mathbb{Z})_{\text{tors}}$ under the Gysin map g, is precisely $\alpha \in H^2(S_1, \mathbb{Z})$. In particular if $\alpha \neq 0$, then $b_{W-W_1} \neq 0$.*

PROPOSITION 5

Let W be a conic bundle of type I over S where $H^1(S, \mathbb{Z}) = 0$ and with $b_W \neq 0$ in $H^3(S, \mathbb{Z})_{\text{tors}}$. Suppose that there exists another topological $\mathbb{P}^1$ – bundle $U \to S$ with the property that $H^3(U, \mathbb{Z})_{\text{tors}} = (0)$. Then $b_W = \pm b_U$ and $H^3(S, \mathbb{Z})_{\text{tors}}$ is generated by b_W.

Proof. To prove this proposition, we shall appeal to the following well-known (cf [11]).

Lemma 2. Let $U \to S$ be a $\mathbb{P}^1$ – bundle over a path connected space S with $H^1(S) = 0$. Then the kernel of the induced homomorphism $H^3(S, \mathbb{Z}) \to H^3(U, \mathbb{Z})$ is generated by b_U.

We now apply the lemma to $U \to S$. Since we have $H^3(U, \mathbb{Z})_{\text{tors}} = (0)$, we get $H^3(S, \mathbb{Z})_{\text{tors}}$ to be generated by b_U, which is a 2-torsion element. Also b_W lies in $H^3(S, \mathbb{Z})_{\text{tors}}$, and $b_W \neq 0$ which implies $b_W = \pm b_U$. This proves Prop. 5.

The next step is to construct explicitly a $\mathbb{P}^1$ – bundle on the subspace $Z - Y$ which satisfies the property of Prop. 5. For this purpose, we elaborate in some detail, what is called the 'Hecke correspondence' of [7], in terms of parabolic bundles as remarked in (*).

Let V be a vector bundle on X of rank 2 and degree 0. Suppose we are given a parabolic structure at a point $x \in X$, defined by a 1-dimensional subspace

$$F^2 V_x \subset F^1 V_x = V_x \quad \text{and weights } (\alpha_1, \alpha_2) \text{ such that}$$

(i) parabolic stable = parabolic semi-stable,
(ii) parabolic stable $\Rightarrow$ underlying bundle is semi-stable, and
(iii) underlying bundle stable $\Rightarrow$ any parabolic structure is stable.

Let T be the torsion $\mathcal{O}_x$-module given by

$$T_x = V_x / F^2 V_x, \quad T_y = 0, \quad x \neq y.$$

(*) Mehta V and Seshadri C S, Moduli of vector bundles on curves with parabolic structures. *Math. Ann.* **248** (1980) 205–239.

Then we have a homomorphism of V onto T (as $\mathcal{O}_X$-modules). If W is the kernel of this map, we have $0 \to W \to V \to T \to 0$ and W is locally free of rank 2 and degree -1.

Let $\tilde{M}$ be the moduli space of parabolic stable bundles of rank 2, degree 0 on X and M_{-1} the moduli space of stable bundles of rank 2, degree -1, $f: \tilde{M} \to M$, the canonical morphism; and $\tilde{M}_0 = f^{-1}(M_0)$.

PROPOSITION 6

If $V \in \tilde{M}$ then W defined above, is in M_{-1} and the map $\psi: \tilde{M} \to M_{-1}$, $V \mapsto W$ is a $\mathbb{P}^1$-bundle, locally trivial in the Zariski topology. In fact it is the dual projective Poincaré bundle on M_{-1}.

Proof. We first claim that if V is parabolic stable then W is stable. To see this, let $F \subset W$ be a line subbundle. We need to show that $\deg F < 0$. Suppose this is not the case i.e. $\deg F \geqslant 0$.

Let G be the line subbundle of V generated by the image of F in V. Then $\deg F \leqslant \deg G$. Since the underlying bundle of V is certainly semi-stable, we have $\deg G \leqslant 0$. By our assumption $\deg F \geqslant 0$ and hence we have $\deg F = \deg G = 0$. This implies that the canonical homomorphism $F \to G$ is an isomorphism. We also see that by the definition of T

$$G_x \subset F^2 V_x,$$

but V being parabolic stable with weights $0 < \alpha_1 < \alpha_2$ we get

$$\text{par deg } G = \alpha_2 < \tfrac{1}{2}(\alpha_1 + \alpha_2) = \text{par deg } V/\text{rk } V$$

which leads to a contradiction. Hence W is stable. Conversely, we claim that $\tilde{M}$ is isomorphic to the dual projective Poincaré bundle of M_{-1} restricted to M_{-1}. To see this, we start with a $W \in M_{-1}$. Then, given a point in $\mathbb{P}(W_x^*)$, $x \in X$, one can easily obtain a vector bundle V of rank 2 and degree 0 and an injection $W \to V$ as $\mathcal{O}_x$-modules. The cokernel then gives a 1-dimensional subspace $F^2 V_x$ of V_x and therefore a 'quasi-parabolic structure'. The stability of W together with an argument as above, makes V parabolic stable. That this map is an *isomorphism* is a consequence of the universal property of the moduli space of parabolic stable bundles.

That $\tilde{M} \to M_{-1}$ is locally trivial in the Zariski topology, now follows from Serre [15].

Q.E.D.

PROPOSITION 7

Consider the canonical morphism $f: \tilde{M}_0 \to M_0$. Then f is a $\mathbb{P}^1$-fibration over M_0^s and $f^{-1}(K)$ has codimension $g-1$ in $\tilde{M}_0$.

Proof. That f is a $\mathbb{P}^1$-fibration over M_0^s is immediate by the property (3) mentioned before Prop. 6. Let $L \oplus L^{-1} \in K - K_0$. Then the points of $\tilde{M}_0$ lying over $L \oplus L^{-1}$ are of the following form:

Case 1. V is a non-trivial extension of L^{-1} by L (or L by L^{-1})

We claim that a parabolic structure on V which is equivalent to giving a subspace $F^2 V_P$ of V_P of dimension one, is stable if and only if $L_P \neq F^2 V_P$. This is necessary to

ensure parabolic stability, for otherwise if $L_P \not\subset F^2 V_P$, then per deg $L = \deg L + \alpha_2 = \alpha_2$ and $\alpha_2 \not< \text{par deg } V/\text{rk } V = \frac{1}{2}(\alpha_1 + \alpha_2)$, since $\alpha_1 < \alpha_2$.

Case 2. $V = L \oplus L^{-1}$

We claim that a parabolic structure $F^2 V_P$ such that $F^2 V_P \neq L_P$ or L_P^{-1} is stable. This is easily checked as above. In fact we see by an argument as in Prop. 1 of [17] all the parabolic structures of Case 2 are isomorphic and hence give one point of M. Hence the total dimension of the fibre at $L \oplus L^{-1} = \dim \text{Ext}(L, L^{-1}) + 1 = g - 1$. Therefore, $\dim f^{-1}(K - K_0) = 2g - 1$.

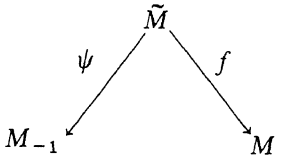
In fact, it is not difficult to see that for $x \in K - K_0$, $f^{-1}(x)$ is the union of two projective spaces of dimension $g - 1$ meeting at a point.

Finally, let $V \in M_0$ be such that $\text{gr } V = L \oplus L$. ($L$ of order two). Then the following can easily be checked.

(i) V has a parabolic stable structure if and only if V is a non-trivial extension of L by L.
(ii) A parabolic structure given by $F^2 V_P$ is stable iff $F^2 V_P \neq L_P$ (where L is the unique line subbundle of V).

Once again by an argument as in Prop. 1 [17] we see that all the parabolic structures on a non-trivial extension V of L by L are isomorphic. Hence the fibre of f over $L \oplus L$ is isomorphic to $\mathbb{P}(H^1(X, \mathcal{O}_x))$ which has dimension $g - 1$, implying $\text{codim}(f^{-1}(K), \tilde{M}_0) = g - 1$.

Remark 6. Thus we have the following diagram



which gives a correspondence between M_{-1} and M.

PROPOSITION 8

The fibration $Y \xrightarrow{p} K - K_0$ *with fibre* $F = \mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$ *satisfies the conditions of the Leray–Hirsch theorem and consequently we have*

$$H^*(Y, \mathbb{R}) \simeq H^*(K - K_0, \mathbb{R}) \otimes H^*(F, \mathbb{R}).$$

Proof. The following form of the Leray–Hirsch theorem will suit our purposes.

Leray–Hirsch. Let E be a fibre bundle over B and compact fibre F. Suppose B has a finite good cover. If there are global cohomology classes $e_1, \ldots, e_r$ on E which, when restricted to each fibre freely, generate the cohomology of the fibre, then $H^*(E, \mathbb{R})$ is a free-module over $H^*(B, \mathbb{R})$ with basis $e_1, \ldots, e_r$; or more precisely, if the canonical map $j: H^*(E, \mathbb{R}) \to H^*(F, \mathbb{R})$, is surjective, then for any subspace W of $H^*(E, \mathbb{R})$ such that $j|W: W \to H^*(F, \mathbb{R})$ is an isomorphism, one has

$$H^*(E, \mathbb{R}) = H^*(B, \mathbb{R}) \otimes W.$$

Since F in our case is $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$, $H^*(F, \mathbb{R})$ is generated by line bundles on F. Therefore it is enough to check that any line bundle on F can be extended to a line bundle on Y.

By Cor. 2, $Y \to K - K_0$ is locally trivial in the Zariski topology. Let L be a line bundle on F, and $U \subset K - K_0$ be the trivializing Zariski open subset. Then L can be obviously extended to a line bundle on $U \times F$, which we continue to denote by L. Since Y is *smooth*, the bundle L on the open subset $U \times F$ of Y can be extended to a line bundle on Y.

Q.E.D.

PROPOSITION 9

The element $\alpha \in H^2(Y, \mathbb{Z})$, associated to the double cover on Y arising from the conic bundle P is non-zero.

Proof. We claim that this double cover on Y is in fact the pull-back of the covering

$$J - \Gamma \to K - K_0.$$

J being the Jacobian of line bundles of deg 0 on X [for notations cf. § 2].

Since this covering is non-split, it follows from Prop. 3, that the double cover on Y is non-split and the covering element in $H^1(Y, \mathbb{Z}/(2))$ is non-zero.

By Prop. 8 and Spanier [19], $H^1(Y, \mathbb{Z}) = 0$. Hence if we consider the cohomology exact sequence for

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/(2) \to 0$$

we get

$$H^1(Y, \mathbb{Z}/(2)) \hookrightarrow H^2(Y, \mathbb{Z}).$$

Since $\alpha \in H^2(Y, \mathbb{Z})$ is the image of the covering element in $H^1(Y, \mathbb{Z}/(2))$, it is non-zero.

Thus to complete the proof of Prop. 9, it is enough to prove the claim.

Fix $t_0 \in X$. Then if $E \in N_0$, one can easily see that E_{t_0} can be identified with right regular representation of $A = \operatorname{End} E$ (see for e.g. Prop. 5 [17]).

Let $E = V \oplus W$ be an element of Y as in Prop. 3. It is easy to see that the scalars in A do not meet V_{t_0} and W_{t_0} under the above identification. So if we consider the projective space $\mathbb{P}(A')$, $A' = A/(\text{scalars})$, then V_{t_0} and W_{t_0} give a pair of lines in $\mathbb{P}(A')$. By Prop. 3, identifying the algebra A with a C_q^+ corresponding to a quadratic form q in Φ, it is clear that this pair of lines is indeed the ones in the conic bundle over Y.

Then the one-dimensional subspaces L_{t_0} and $L_{t_0}^{-1}$ give a pair of points $\bar{L}_{t_0}$ and $\bar{L}_{t_0}$ in $\mathbb{P}(A')$. Then the correspondence

$$E \mapsto (\bar{L}_{t_0}, \bar{L}_{t_0}^{-1})$$

gives a double covering on Y since we have a defining family of vector bundles $E_y = \{V_y \oplus W_y\}_{y \in Y}$. Obviously, this is the canonical double cover associated to the conic bundle on Y.

Note that $\{L_y \oplus L_y^{-1}\}_{y \in Y}$ gives a family on Y which is clearly the pull-back $p^*\{L_u \oplus L_u^{-1}\}_{u \in K - K_0}$, under $p: Y \to K - K_0$.

The double cover of Y given above is therefore the pull-back of the double cover of $K - K_0$ given by $J - \Gamma \to K - K_0$.

Q.E.D.

PROPOSITION 10

(a) *Let Z and Y be as in* §3. *Then there exists a topological $\mathbb{P}^1$ – bundle D on $Z - Y$ with $H^*(D, \mathbb{Z})$ torsion free. In fact $D = f^{-1}(M_0^s)$.*
(b) *The topological Brauer class $b_D \neq 0$.*

Proof. (a) By Prop. 7, $f^{-1}(K)$ has codimension $g-1$ in $\tilde{M}_0$ and $D = \tilde{M}_0 - f^{-1}(K)$.

Consider $\psi : \tilde{M}_0 \to M_{-1,x}, M_{-1,x}$ being bundles in M_{-1} with determinant L_x. Since the $\mathbb{P}^1$ fibration ψ is locally trivial in the Zariski topology, a line bundle L on the fibre $\mathbb{P}^1$ can be extended obviously to $\mathbb{P}^1 \times U$, where U is a Zariski open subset of $M_{-1,x}$. Since $\tilde{M}_0$ is smooth, the closure of L in $\tilde{M}_0$ gives a line bundle on $\tilde{M}_0$. Now, the cohomology of $\mathbb{P}^1$ is generated by line bundles and therefore we can apply Leray–Hirsch theorem to conclude that the cohomology groups of $\tilde{M}_0$ are those of $\mathbb{P}^1 \times M_{-1,x}$.

By Atiyah–Bott [2], all the cohomology groups of $M_{-1,x}$ are torsion-free and therefore all the cohomology groups of $\tilde{M}_0$ are also torsion-free.

Since $g \geqslant 3$, the complex codimension of $f^{-1}(k)$ in $\tilde{M}_0 = g - 1 \geqslant 2$. This implies $\mathrm{Codim}_{\mathbb{R}} f^{-1}(K)$ in $\tilde{M}_0 \geqslant 4 = g - 1 \geqslant 2$.

Consider the homology sequence of the pair $(\tilde{M}_0, D)$

$$H_k(\tilde{M}_0, D, \mathbb{Z}) \to H_{k-1}(D, \mathbb{Z}) \to H_{k-1}(\tilde{M}_0, \mathbb{Z}) \to H_{k-1}(\tilde{M}_0, D, \mathbb{Z})$$

$\tilde{M}_0$ is a compact complex manifold and so we can apply Alexander duality to the pair $(\tilde{M}_0, D)$ to get

$$\begin{aligned} H_k(\tilde{M}_0, D, \mathbb{Z}) &\simeq H^{n-k}(\tilde{M}_0 - D; \mathbb{Z}) \\ &= H^{n-k}(f^{-1}(K), \mathbb{Z}) \\ n &= \dim_{\mathbb{R}} \tilde{M}_0. \end{aligned}$$

Since $\dim_{\mathbb{R}} f^{-1}(K) \leqslant n - 4$, we therefore get

$$H_2(\tilde{M}_0, D; \mathbb{Z}) = H^{n-2}(f^{-1}(K), \mathbb{Z}) = 0$$

and similarly $H_3(\tilde{M}_0, D, \mathbb{Z}) = 0$.

$$H_2(D, \mathbb{Z}) = H_2(\tilde{M}_0, \mathbb{Z}).$$

By the 'universal coefficient theorem' one has torsion subgroup of $H_k(T, \mathbb{Z})$ to be that of $H^{k+1}(T, \mathbb{Z})$, T any topological space, and therefore we conclude that

$$H^3(D, \mathbb{Z})_{\mathrm{tors}} = H^3(\tilde{M}_0, \mathbb{Z})_{\mathrm{tors}} = (0).$$

Note that $Z - Y = M_0^s$ and this completes the proof.

Q.E.D.

The claim (b) is due to Ramanan (p. 52 [18])

Theorem 3. *$H^3(Z, \mathbb{Z})$ is torsion free.*

Proof. Consider the Gysin sequence for $(Z, Z-Y)$,

$$H^1(Y,\mathbb{Z}) \to H^3(Z,\mathbb{Z}) \to H^3(Z-Y,\mathbb{Z}) \xrightarrow{g} H^2(Y,\mathbb{Z})$$

Now by Cor. 2, Y is a $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$ fibration over $K-K_0$ and by ([19] p. 159) $H^1(K-K_0,\mathbb{Z})=0$ implying by standard arguments $H^1(Y,\mathbb{Z})=0$ (note that $H^1(Y,\mathbb{Z})$ is torsion-free by the universal coefficient theorem).

Thus we have from the Gysin sequence an injection

$$H^3(Z,\mathbb{Z}) \hookrightarrow H^3(Z-Y,\mathbb{Z}). \qquad (*)$$

Now note that $H^1(Z-Y,\mathbb{Z})=0$. (This follows for example from the Gysin sequence. For, note that $H^1(Z-Y,\mathbb{Z}) \simeq H^1(Z,\mathbb{Z}.)$ Also we will be seeing in §5 that the codimension of N_0-Z in N_0 is actually 6. But N_0 is unirational and is therefore simply connected, being smooth projective (cf. Serre [16]). Hence $H^1(N_0,\mathbb{Z})=0$ implying $H^1(Z,\mathbb{Z})=0=H^1(Z-Y,\mathbb{Z})$.

Thus we can now apply Prop. 5 and Prop. 10 to see that $H^3(Z-Y,\mathbb{Z})_{\text{tors}}$ is generated by b_{P-P_1}, the Brauer element coming from the conic bundle $P-P_1$ over $N_0-N_1=Z-Y$. By Prop. 4 the total space of $P-P_2$ is smooth and hence the theorem due to Nitsure mentioned in §4·1 is applicable. Thus we have

$$g(b_{P-P_1})=\alpha \neq 0 \quad (\alpha \neq 0 \text{ by Prop. 9}).$$

This together with (*) and the exactness of the Gysin sequence gives $H^3(Z,\mathbb{Z})_{\text{tors}}=(0)$ Q.E.D.

Lemma 3. $\operatorname{Pic} Z$ *is generated by* $\operatorname{Pic}(Z-Y)$ *and the element* $[Y]$ *coming from the irreducible divisor* $Y \subset Z$.

Proof. This follows from the following general fact:

If X is a smooth variety, $U \subset X$ open with $Y=X-U$ an irreducible divisor, then

$$\operatorname{Pic} X \to \operatorname{Pic} U$$

is a surjection and the kernel of this homomorphism is generated by $[Y]$.

Lemma 4. Let $N_1 \subset N_0$ *be as in* §3. *Then* $\operatorname{Pic} N_0$ *is generated by* $\operatorname{Pic} M_0$ *and* $[N_1]$ *over* $\mathbb{Q}$(*).

Proof. Firstly, we remark that N_1 is precisely $\overline{Y}$ in N_0. Actually, we will be showing in §5 that $Y \subset N_1$ is precisely the set of non-singular points of N_1. Let us assume this. Suppose N_1 is not irreducible and let A, B be subvarieties such that $N_1=A \cup B$. Then $A \cap B \subset N_1-Y$ and hence $A \cap Y$ and $B \cap Y$ will disconnect Y which

(*) In fact, over $\mathbb{Z}$ (see Remark in Appendix 2).

is false since Y is connected. Thus N_1 is irreducible. Also since Y is irreducible it follows that $\bar{Y}=N_1$.

An application of Lemma 4 and the result of Appendix 2 yields our result.

Remark 7. Thus by the above lemma, any $L\in\operatorname{Pic} N_0$ can be expressed as $L=aL_1+bL_2$, $L_1=[N_1]$ and $L_2\in\operatorname{Pic} M_0$, $a, b\in\mathbb{Q}$.

In particular, let L be chosen ample. Then if F is the fibre of $Y\to K-K_0$, L when restricted to F is $(aL_1+bL_2)|F$. But since $L_2\in\operatorname{Pic} M_0$, which is trivial on F, we have

$$L|F=(aL_1)|F$$

F is $\mathbb{P}^{g-2}\times\mathbb{P}^{g-2}$ and L is ample, therefore we have the restriction of L_1 to each $\mathbb{P}^{g-2}$ to be either ample or negatively ample.

Let $e\in H^2(Y,\mathbb{R})$ be the Euler class of the irreducible divisor Y in Z. Then by the 'adjunction formula', we have

$$e=[Y]|_Y,$$

where $[Y]$ is the class of $Y\subset Z$. Now $L_1=[N_1]$ and $N_1=\bar{Y}$, hence it follows from the above reasoning that *the Euler class e when restricted to the factors of F is ample or negatively ample.*

PROPOSITION 11

Let E be the normal bundle of Y in Z and E_0 be the compliment of the zero section. Consider the Gysin sequence for the 2-plane bundle (E, E_0)

$$H^k(Y,\mathbb{R})\to H^{k+2}(Y,\mathbb{R})\to H^{k+2}(E_0,\mathbb{R})\to H^{k+1}(Y,\mathbb{R})\to H^{k+3}(Y,\mathbb{R}).$$

Then the Gysin homomorphism

$$h: H^k(Y,\mathbb{R})\to H^{k+2}(Y,\mathbb{R}),$$

given by 'wedging' with the Euler class $e\in H^2(Y,\mathbb{R})$ is an injection for $k\leqslant \dim_{\mathbb{R}}\mathbb{P}^{g-2}-2=2g-6$.

Proof. By Prop. 8 we have

$$H^k(Y)\simeq\sum_{l+m=k}H^l(K-K_0)\otimes H^m(F)$$

or using the subspace W of $H^*(Y)$ as in Prop. 3.8, we have, any $u\in H^k(Y)$ $u\neq 0$ and $k\leqslant\dim_{\mathbb{R}}F$, to be expressible as

$$v=\sum_i u_i\otimes w_i,\quad u_i\in H^*(K-K_0),\quad w_i\in W,$$

where not all $w_i=0$ (this is so since $k\leqslant\dim_{\mathbb{R}}F$). Without loss of generality, the u_i's can be chosen linearly independent.

Now consider $u\otimes e$, e the Euler class in $H^2(Y,\mathbb{R})$

$$u\otimes e=\sum_i u_i\otimes(w_i\otimes e).$$

Consider the class $w_i \otimes e$. This when restricted to the fibre F is non-zero, since by Remark 7, the class e restricted to the factors of F is ample or negatively ample and w_i by definition lies in W and so $w_i \wedge e$ is non-zero on F for $w_i \in H^k(F, \mathbb{R})$, $k \leqslant \dim_{\mathbb{R}} \mathbb{P}^{g-2} - 2$. Hence by the linear independence of the u_i's we get

$$u \otimes e = \sum_i u_i \otimes (w_i \otimes e) \neq 0$$

$\Rightarrow$ $h: H^k(Y, \mathbb{R}) \to H^{k+2}(Y, \mathbb{R})$ is an injection for $k \leqslant \dim_{\mathbb{R}} \mathbb{P}^{g-2} - 2 = 2g - 6$.

COROLLARY 3

The Gysin map considered in Theorem 3 i.e.

$$h^1: H^k(Y, \mathbb{R}) \to H^{k+2}(Z, \mathbb{R})$$

is also an injection for $k \leqslant 2g - 6$.

Proof. In fact, the Gysin sequences for (E, E_0) and $(Z, Z - Y)$ are related as follows.

$$\begin{array}{ccc} H^k(Y, \mathbb{R}) & \xrightarrow{\quad h \quad} & H^{k+2}(Y, \mathbb{R}) \\ & {\scriptstyle h'} \searrow \quad \nearrow {\scriptstyle \mathrm{Res}} & \\ & H^{k+2}(Z, \mathbb{R}) & \end{array}$$

and therefore, since h is an injection by Prop. 11, so is h^1..

COROLLARY 4

$H^k(Z, \mathbb{R}) = H^{k-2}(Y, \mathbb{R}) \oplus H^k(Z - Y, \mathbb{R})$ $k \leqslant 2g - 4$.

Proof. Consider the Gysin sequence for $(Z, Z - Y)$.

$$\to H^{k-2}(Y, \mathbb{R}) \to H^k(Z, \mathbb{R}) \to H^k(Z - Y, \mathbb{R}) \to H^{k-1}(Y, \mathbb{R}) \to H^{k+1}(Z, \mathbb{R}) \to$$

Since h' is an injection for $k \leqslant 2g - 6$, we get

$$0 \to H^{k-2}(Y, \mathbb{R}) \to H^k(Z, \mathbb{R}) \to H^k(Z - Y, \mathbb{R}) \to 0$$

for $k \leqslant 2g - 4$ and this proves the corollary.

Remark 8. By Kirwan [5], the Betti numbers of M_0^s are known if genus $g \geqslant 4$, for $i < 2g - 3$. This together with Prop. 8, Cor. 4 and Spanier [19], yields the Betti numbers of Z for $i < 2g - 3$.

Remark 9. Let us assume $g \geqslant 4$ and recall from Prop. 10, we had a topological $\mathbb{P}^1$ – bundle D on $Z - Y$. By the proof of Prop. 10 we see that if $g \geqslant 4$, then $\operatorname{codim}_{\mathbb{R}} f^{-1}(K)$ in $M_0 \geqslant 6$ and hence

$$H_k(D, \mathbb{Z}) = H_k(\tilde{M}_0, \mathbb{Z}) \quad \text{for } k \leqslant 4.$$

The homology groups of $\tilde{M}_0$ are known by [10] or by using Atiyah–Bott [2] for $M_{-1,x}$. In particular, rank of $H_3(\tilde{M}_0, \mathbb{Z})$ is $2g$ and hence rank of $H_3(D, \mathbb{Z})$ is $2g$.

We have already seen that $H^1(Z - Y, \mathbb{R}) = 0$. Now D is a $\mathbb{P}^1$-fibration over $Z - Y$ and $H^1(\mathbb{P}^1, \mathbb{R}) = 0, H^1(Z - Y, \mathbb{R}) = 0$. Therefore by the Serre sequence of this fibration (see for example Spanier *Algebraic topology* pp. 519) we get an exact sequence

$$H_3(\mathbb{P}^1, \mathbb{R}) \to H_3(D, \mathbb{R}) \to H_3(Z - Y, \mathbb{R}) \to H_2(\mathbb{P}^1, \mathbb{R}) \to H_2(D, \mathbb{R}) \to H_2(Z - Y, \mathbb{R}) \to H_1(\mathbb{P}^1, \mathbb{R}).$$

Now, $H_3(\mathbb{P}^1, \mathbb{R}) = H_1(\mathbb{P}^1, \mathbb{R}) = 0, H_2(\mathbb{P}^1, \mathbb{R}) \simeq \mathbb{R}$. Thus we have

$$0 \to H_3(D, \mathbb{R}) \to H_3(Z - Y, \mathbb{R}) \to H_2(\mathbb{P}^1, \mathbb{R}) \to H_2(D, \mathbb{R})$$

$H_2(Z - Y, \mathbb{R}) \to 0$. By the Picard group computations it follows that, $H_2(D, \mathbb{R}) = \mathbb{R}^2$ and $H_2(Z - Y, \mathbb{R}) = \mathbb{R}$, and therefore we have

$$\text{rank of } H_3(Z - Y, \mathbb{R}) = \text{rank } H_3(D, \mathbb{R}) = 2g.$$

Thus the rank of $H_3(Z - Y, \mathbb{R}) = 2g$ and hence the rank of $H^3(Z - Y, \mathbb{R})$ is $2g$.

Theorem 4. $H^3(Z, \mathbb{Z}) = \mathbb{Z}^{2g}$, *when* $g \geqslant 4$.

Proof. By Theorem 3 $H^3(Z, \mathbb{Z})$ is torsion-free. By Cor. 4

$$H^3(Z, \mathbb{R}) = H^1(Y, \mathbb{R}) \oplus H^3(Z - Y, \mathbb{R})$$

Since $H^1(Y, \mathbb{R}) = 0$, using Remark 9 we conclude that $H^3(Z, \mathbb{Z}) = \mathbb{Z}^{2g}$.

5. The main theorem

Consider the stratification of N_0 in terms of the degeneracy locus as in § 3, $N_3 \subset N_2 \subset N_1 \subset N_0$.

PROPOSITION 12

The subvariety N_2 has codimension 3 in N_0.

Proof. Consider the local morphism

$$\varphi: N_0 \to \mathscr{A}$$

of § 2. We have already seen that $\varphi: N_1 \to \mathscr{A}_1$ and $\varphi: N_2 \to \mathscr{A}_2$. Moreover, φ being a smooth local morphism, its fibres are equidimensional. Hence the codimension of N_2 in N_0 equals the codimension of $\mathscr{A}_2$ in $\mathscr{A}$. We have also seen that $\mathscr{A}_1 \subset \mathscr{A}$ is a hypersurface given by $\Delta = 0$ and $\mathscr{A}_2 \subset \mathscr{A}_1$ is precisely the singular locus of $\mathscr{A}_1$. So we would like to show that

$$\text{codim of } \mathscr{A}_2 \text{ in } \mathscr{A}_1 = 2.$$

Consider the natural conic bundle C on $\mathbb{A}^6$ as in Lemma 1. Let S be the hypersurface of $\mathbb{A}^6$ given by $\Delta = 0$ and let $S^1 \subset S$ be its singular locus. Then by Remark 5, it is

enough to show that

$$\text{codim of } S^1 \text{ in } S = 2.$$

By definition, if

$$q = aX^2 + bY^2 + cZ^2 + fYZ + gXZ + hXY,$$

then Δ is given by

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Thus, if $\mathrm{Sym}(\cdot /\!\!/_3)$ is all (3×3)-symmetric matrices

$$S = \{A \in \mathrm{Sym}(\cdot /\!\!/_3) | \text{rank } A \leqslant 2\}.$$

The conditions $\partial\Delta/\partial a = \partial\Delta/\partial b = \partial\Delta/\partial c = \partial\Delta/\partial f = \partial\Delta/\partial g = \partial\Delta/\partial h = 0$, gives

$$bc = f^2, \quad ac = g^2, \quad ab = h^2, \quad af = hg, \quad fh = bg, \quad ch = fg.$$

$$a/h = h/b = g/f \quad \text{and} \quad a/g = h/f = g/c$$

i.e. $$S^1 = \{A \in \mathrm{Sym}(\cdot /\!\!/_3) | \text{rank } A \leqslant 1\}.$$

From which we obtain the codim of S^1 in S.

Q.E.D.

COROLLARY 5

$H_k(N_0, \mathbb{Z}) = H_k(Z, \mathbb{Z}), k \leqslant 4.$

Proof. Consider the homology sequence of the pair (N_0, Z)

$$H_{k+1}(N_0, Z; \mathbb{Z}) \to H_k(Z, \mathbb{Z}) \to H_k(N_0, \mathbb{Z}) \to H_k(N_0, Z; \mathbb{Z}).$$

Since N_0 is a compact complex manifold, the Alexander duality as in Theorem 3, gives

$$H_k(N_0, Z, \mathbb{Z}) \simeq H^{n-k}(N_0 - Z, \mathbb{Z}) = H^{n-k}(N_2, \mathbb{Z}).$$

$$n = \dim_{\mathbb{R}} N_0.$$

By Prop. 12, $\dim_{\mathbb{R}} N_2 = n - 6$ since $\mathrm{codim}_{\mathbb{C}}(N_2, N_0) = 3$. Hence $H^{n-k}(N_2, \mathbb{Z}) = 0$ for $k < 6$.

$$\Rightarrow \quad H_k(N_0, Z, \mathbb{Z}) = 0 \quad k < 6$$

$$\Rightarrow \quad H_k(N_0, \mathbb{Z}) = H_k(Z, \mathbb{Z}), \quad k \leqslant 4.$$

Theorem 5. $H^3(N_0, \mathbb{Z}) = \mathbb{Z}^{2g}$.

Proof. Firstly, $H^3(N_0, \mathbb{Z})$ is torsion-free. For, by Cor. 5, $H_2(N_0, \mathbb{Z}) = H_2(Z, \mathbb{Z})$ and

therefore by the universal coefficient theorem, since

$$H^3(N_0, \mathbb{Z})_{\text{tors}} = H_2(N_0, \mathbb{Z})_{\text{tors}},$$

we have

$$H^3(N_0, \mathbb{Z})_{\text{tors}} = H^3(Z, \mathbb{Z})_{\text{tors}} = (0) \quad \text{by Theorem 3.6.}$$

Now using Theorem 4 and for Cor. 5 we get

$$H^3(N_0, \mathbb{Z}) = \mathbb{Z}^{2g}. \qquad \text{Q.E.D.}$$

Theorem 6. *The Betti number B_4 of N_0 is* $B_4(N_0) = \binom{2g}{2} + 4$.

Proof. To see this, we use Prop. 5.9 and Remark 5.11 of Kirwan [5] to get the Betti numbers of M_0^s as $B_0 = 1$, $B_1 = 0$, $B_2 = 1$, $B_3 = 2g$, $B_4 = 2$, etc.

By Cor. 4,

$$B_4(Z) = B_2(Y) + B_4(Z - Y).$$

Now, by Prop. 8, $B_2(Y) = B_2(K - K_0) + B_2(\mathbb{P}^{g-2} \times \mathbb{P}^{g-2})$. Hence, by Spanier [19]

$$B_2(Y) = \binom{2g}{2} + 2.$$

Also, $B_3(Y) = 0$, since the odd Betti numbers of $K - K_0$ and $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$ are zero (cf. [19] again). Combining this with (*), we get

$$B_4(Z) = \binom{2g}{2} + 4.$$

Hence by Cor. 5 we get

$$B_4(N_0) = \binom{2g}{2} + 4.$$

Q.E.D.

Appendix 1

We present here a proof due to Coliot–Thélène of Theorem (A) mentioned in the introduction. We shall make a few remarks before going into the proof.

Let X be a smooth variety over $\mathbb{C}$. For the notations and properties of most of the facts mentioned below (cf. Grothendieck [4] and Saltman [13], [14]).

Define $\text{Br}(X)$ to be the Brauer group of Azumaya algebras on X. Let $\text{Br}'(X)$ be the 'cohomological Brauer group' of X defined to be $H^2_{\text{et}}(X, \mathbb{G}_m)_{\text{tor}}$. Then the following facts are well known:

(i) $\text{Br}\, X$ is contained in $\text{Br}'(X)$.
(ii) If X is a unirational smooth proper variety, then $\text{Br}'(X) = H^3(X(\mathbb{C}), \mathbb{Z})_{\text{tor}}$.
(iii) Define $\text{Br}_{nr}(X)$, the unramified Brauer group of X to be $\text{Br}_{nr}(X) = \text{Br}'(\bar{X})$, $\bar{X}$ any

smooth compactification of X. Then it is know that $\mathrm{Br}_{nr}(X)$ is independent of the choice of $\bar{X}$ since we are in characteristic 0.

(iv) Another way of defining $\mathrm{Br}_{nr}(X)$ is as follows: Let $\mathbb{C}(X)$ be the function field of X. Then for every discrete valuation ring A, with $\mathbb{C} \subset A \subset \mathbb{C}(X)$, and quotient field of $A = \mathbb{C}(X)$, there exists a natural homomorphism

$$\partial_A : \mathrm{Br}\,\mathbb{C}(X) \to H^1(\mathbb{K}_A, \mathbb{Q}/\mathbb{Z}).$$

$\mathbb{K}_A$–the residue class field of A.

Define

$$\mathrm{Br}_{nr}\,\mathbb{C}(X) = \bigcap_{\text{all such } A} (\mathrm{Ker}\,\partial_A) \quad \text{and} \quad \mathrm{Br}_{nr}\,X = \mathrm{Br}_{nr}\,\mathbb{C}(X).$$

(v) Let k be a field and C a conic over k, i.e. a conic bundle coming from a quaternion algebra over k. Then there is a canonical homomorphism

$$\mathrm{Br}'(k) \to \mathrm{Br}'(C)$$

and the kernel of this homomorphism is the 2-torsion element coming from the quaternion algebra over k associated to C.

Note that for a field k, $\mathrm{Br}'(k) = \mathrm{Br}(k)$.

PROPOSITION 13

Let C be a conic bundle on X with $\mathrm{Br}'(C) = 0$, and let η be the generic point of X. Let C_η be the restriction of C over $\mathbb{C}(\eta)$. To C_η we associate an element $\alpha_\eta \in \mathrm{Br}\,\mathbb{C}(\eta)$. Suppose that for the conic bundle C on X, there exists a discrete valuation ring A, with quotient field of $A = \mathbb{C}(X)$, $\mathbb{C} \subset A \subset \mathbb{C}(X)$, such that $\partial_A(\alpha_\eta) \neq 0$. Then $\mathrm{Br}_{nr}(X) = 0$.

Proof. Suppose that $\mathrm{Br}_{nr}(X) \neq 0$ and let $\alpha \in \mathrm{Br}_{nr}(X) = \mathrm{Br}_{nr}(\mathbb{C}(X))$ be a non-zero element. Consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{Br}_{nr}\,X & \longrightarrow & \mathrm{Br}_{nr}\,C \\ \downarrow & & \downarrow \\ \mathrm{Br}'\,\mathbb{C}(\eta) & \longrightarrow & \mathrm{Br}'\,C_\eta \end{array}$$

where the map $\mathrm{Br}_{nr}(X) \to \mathrm{Br}_{nr}(C)$ is the canonical map induced from $C \to X$ and the vertical maps are

$$\mathrm{Br}_{nr}\,C \hookrightarrow \mathrm{Br}\,C \hookrightarrow \mathrm{Br}'\,C \to N\,\mathrm{Br}'\,C_\eta$$

$$\mathrm{Br}_{nr}\,X \hookrightarrow \mathrm{Br}\,\mathbb{C}(X) = \mathrm{Br}\,\mathbb{C}(\eta) = \mathrm{Br}'\,\mathbb{C}(\eta).$$

Consider the image of α in $\mathrm{Br}'\,\mathbb{C}(\eta)$, call it α_η. Then since $\mathrm{Br}'\,C = 0$, the above diagram gives

$$\alpha_\eta \in \mathrm{Ker}\,[\mathrm{Br}'\,\mathbb{C}(\eta) \to \mathrm{Br}'\,C_\eta]$$

and therefore by Remark (v), α_η is the element in $\mathrm{Br}\,\mathbb{C}(\eta)$ associated to the conic C_η. Now by the hypothesis of the proposition, there exists a discrete valuation ring

$A, \mathbb{C} \subset A \subset \mathbb{C}(X)$ with quotient field of $A = \mathbb{C}(X)$, such that

$$\partial_A(\alpha_\eta) \neq 0. \qquad (*)$$

But $\alpha \in \mathrm{Br}_{nr}\mathbb{C}(X)$ and $\mathrm{Br}_{nr}\mathbb{C}(X)$ is by definition equal to

$$\bigcap_{\text{all such } A} (\mathrm{Ker}\,\partial_A),$$

implying

$$\partial_A(\alpha_\eta) = 0$$

which contradicts (*). Hence the proposition.

Q.E.D.

Now let us consider the variety M_0^s, the moduli space of stable vector bundles of rank 2 and trivial determinant. Then by Prop. 3, there is a conic bundle D on M_0^s with $H^3(D, \mathbb{Z})_{\mathrm{tors}} = (0)$ and therefore $\mathrm{Br}'(D) = 0$.

The existence of an A with the requisite properties of the Prop. 13 is precisely the theorem due to Nitsure [9]. Indeed, in the notation of §4, the irreducible divisor $Y \subset Z$ provides us with the discrete valuation ring A.

Hence by Prop. 13, $\mathrm{Br}_{nr}M_0^s = 0$. This implies by Remark (3), that $\mathrm{Br}'(N_0) = 0$, since N_0 is a smooth compactification of M_0^s. Now N_0 is unirational, smooth-projective and therefore by Remark (ii), $\mathrm{Br}'(N_0) = H^3(N_0, \mathbb{Z})_{\mathrm{tor}} = (0)$.

Appendix 2

Theorem (*C S Seshadri*). *Let M be the moduli space of semi-stable vector bundles of rank 2^n and degree d. Then*

$$\mathrm{Pic}\,M^s \text{ (as well as } \mathrm{Pic}\,M) \simeq \mathbb{Z}.$$

Proof. For simplicity we present the proof only for rank 2 and degree zero. Choose m such that for all stable bundles V of rank two and degree zero, $V(m)$ is generated by the global sections. Then if E denotes the trivial vector bundle of rank $r = \dim H^0(V(m))$, $V(m)$ is canonically a quotient of E and $V(m)$ represents a point of $Q = Q(E/P)$, the Quot scheme of quotients of E with Hilbert polynomial equal to P.

We then have an open subscheme Q^s of Q representing quotient vector bundles W of E such that W is stable and the canonical homomorphism $H^0(E) \to H^0(W)$, is an isomorphism. Thus we have a canonical morphism

$$p: Q^s \to M_1^s,$$

where M_1^s is the moduli space of stable vector bundles of rank 2 and $\det = \mathcal{O}_X(2m)$ and p is a G-principal fibre space with $G = PGL(H^0(E))$. Note that $M_1^s \approx M_0^s$ of §2.

Let $q: B \to M_1^s$ be the fibre space associated to p with fibre the projective space of dimension $(r-1)$. Hence if $W \in M_1^s$, the fibre $q^{-1}(W)$ can be canonically identified with $\mathbb{P}(H^0(W))$.

Let A denote the projective space $\mathbb{P}(\mathrm{Ext}(L, I))$, the 'Atiyah family' on the vector

space of all extensions of the form

$$0 \to I \to W \to L \to 0,$$

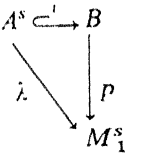
where I is the trivial vector bundle of rank one and L the line bundle $\mathcal{O}_X(2m)$. Let A^s denote the subset of A defined by

$$A^s = \{0 \to I \to W \to L \to 0 \mid W \text{ is stable}\}.$$

Then A^s is open and we have a canonical surjective morphism

$$\lambda : A^s \to M_1^s$$

which associates to an extension as above the vector bundle W. Observe that giving an extension as above is equivalent to giving a section $s \in H^0(W)$ which is non-vanishing at every point $x \in X$. From this observation we deduce easily that A^s can be identified canonically as an open subset of the projective bundle B over M_1^s; in fact we have a commutative diagram



Note that $p^{-1}(W) - \lambda^{-1}(W)$ is *irreducible* in $\mathbb{P}(H^0(W))$ for $p^{-1}(W) - \lambda^{-1}(W)$ is the canonical image in $\mathbb{P}(H^0(W))$ of the set $S = \{s \mid s \in H^0(W), s \text{ vanishes at least at one point of } X\}$ i.e.

$$S = \bigcup_{x \in X} \operatorname{Ker}(H^0(W) \to W_x).$$

Since $\lambda^{-1}(W)$ is the complement of an irreducible closed subset in a projective space, nonvanishing regular functions on $\lambda^{-1}(W)$ reduce to constants. From this, we easily conclude that, if U is an open subset in M_1^s and f a regular nonvanishing function on $\lambda^{-1}(U)$, then f is a pull-back of a regular nonvanishing function on U. From these properties, it follows easily that the canonical homomorphism

$$\lambda^* : \operatorname{Pic} M_1^s \to \operatorname{Pic} A^s$$

is injective. To see this, let $L \in \ker \lambda^*$. Then if L is given by transition functions $\{\theta_{ij}\}$ on $V_i \cap V_j$, we have nonvanishing regular functions φ_i on $\lambda^*(V_i)$ such that $\lambda^*(\theta_{ij}) = \varphi_i \varphi_j^{-1}$.

Now the φ_i are pull-backs of functions θ_i on V_i and the required assertion follows.

Now A^s is an open subset of $\mathbb{P}(\operatorname{Ext}(L, I))$ and therefore $\operatorname{Pic} A^s$ is either $\mathbb{Z}$ or $\mathbb{Z}/(m)$, $m \neq 0$. Consider the normal projective variety M_1. The ample line bundle on M_1 restricted to M_1^s shows that $\operatorname{Pic} M_1^s$ is not torsion. But $\lambda^* : \operatorname{Pic} M_1^s \to \operatorname{Pic} A^s$ is injective implying, $\operatorname{Pic} A^s = \mathbb{Z}$ and also $\operatorname{Pic} M_1^s = \mathbb{Z}$. Since M_1 is normal, it follows that $\operatorname{Pic} M_1 \subset \operatorname{Pic} M_1^s$ and hence $\operatorname{Pic} M_1 = \mathbb{Z}$.

Remark. A priori, $\operatorname{Pic} M_1$ is just a proper subgroup of $\operatorname{Pic} M_1^s$. But if M_1 is locally

factorial then $\mathrm{cl}\, M_1 = \mathrm{Pic}\, M_1$ and we would have $\mathrm{Pic}\, M_1 = \mathrm{Pic}\, M_1^s$.

In fact, this is so and has been recently proved by J M Drezet and M S Narasimhan.

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On a class of slowly changing functions

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Abstract. We study the slowly changing functions and use them in the study of entire functions.

Keyword. Slowly changing functions.

1. Introduction

A function $L(r)$ which satisfies $L(kr) \sim L(r)$ for each positive integer k is called a slowly changing function. Further if $L(r)$ is differentiable then

$$\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0.$$

See [2]. Thus, for instance, $\log r$ is a slowly changing function. We observe that $\log\log r$ is also a slowly changing function. In fact in this case we can say much more viz. if $L(r) = \log\log r$ then

$$\lim_{r\to\infty} \frac{r\log rL'(r)}{L(r)} = 0.$$

In this paper we study such functions which we call the very slowly changing function and use them in the study of entire functions. We start with

DEFINITION

Let $J(r)$ be a function which is positive and differentiable for $r \geqslant r_0$. Then $J(r)$ will be called a very slowly changing function if

$$\lim_{r\to\infty} \frac{r\log rJ'(r)}{J(r)} = 0.$$

Remark. (i) Clearly every very slowly changing function is a slowly changing function and $L(r) = \log r$ shows that the converse need not be true.

(ii) Following Levin [1, 32], a function $\rho(r)$ satisfying the condition $\lim_{r\to\infty} \rho(r) = \rho$ and $\lim_{r\to\infty} r\rho'(r)\log r = 0$ is called a proximate order. Thus we see that every proximate order is a very slowly changing function.

(iii) Throughout this paper $J(r)$ will denote a very slowly changing function.

We first prove some properties of very slowly changing functions.

Theorem 1. *Let $J(r)$ be a very slowly changing function, then*

(i) $J(r^k) \sim J(r)$ *as* $r \to \infty$ *uniformly in* $0 < a \leqslant k \leqslant b < \infty$.
(ii) *For every* $\alpha > 0$ *and* $r \geqslant r_0$*, $J(r)(\log r)^\alpha$ is strictly increasing and* $\to \infty$ *as* $r \to \infty$.
(iii) *For every real* $k \geqslant 0$, $\{[(\log r)^k J(r)]/r\} \to 0$ *as* $r \to \infty$.
(iv) *As* $r \to \infty$

$$\int_{r_0}^{r} \frac{(\log t)^\alpha J(t)}{t}\,dt \sim \frac{J(r)(\log r)^{\alpha+1}}{\alpha+1} \quad \textit{for} \quad \alpha > -1.$$

(v) *As* $r \to \infty$

$$\int_{r_0}^{r} \frac{J(t)(\log t)^{-\beta}}{t}\,dt \sim \frac{J(r)}{(\beta-1)(\log r)^{\beta-1}} \quad \textit{for} \quad \beta > 1.$$

Proof. We first prove (i).

Without any loss of generality we assume $a < 1$, and $b > 1$. Now by hypothesis, given any $\varepsilon > 0$, there exists $r_0 > 0$ such that

$$-\varepsilon < \frac{r \log r J'(r)}{J(r)} < \varepsilon \quad \text{for all} \quad r \geqslant r_0.$$

Let $t \in [a, b]$. Choose r such that $r^a \geqslant r_0$. Then

$$-\varepsilon < \frac{r^t \log (r^t) J'(r^t)}{J(r^t)} < \varepsilon.$$

And so

$$-\varepsilon \int_a^k dt < \int_a^k \frac{r^t (t \log r) J'(r^t)}{J(r^t)}\,dt < \varepsilon \int_a^k dt. \tag{1}$$

But

$$\int_a^k \frac{r^t (t \log r) J'(r^t)}{J(r^t)}\,dt = r \log r \int_a^k \frac{t r^{t-1} J'(r^t)}{J(r^t)}\,dt$$

$$= r \log r \left\{ \log \frac{J(r^k)}{J(r^a)} \right\}.$$

And so from (1) we get

$$-\varepsilon(b-a) \leqslant -\varepsilon(k-a) < r \log r \left\{ \log \frac{J(r^k)}{J(r^a)} \right\} < \varepsilon(k-a) \leqslant \varepsilon(b-a).$$

It now easily follows that

$$\lim_{r \to \infty} \log \frac{J(r^k)}{J(r^a)} = 0.$$

Thus

$$J(r^k) \sim J(r^a) \quad \text{for all} \quad k \in [a, b]. \tag{2}$$

---

* r_0 need not be the same at every place.

In particular, taking $k = 1$ we get $J(r) \sim J(r^a)$. And so, from (2) we get

$$J(r^k) \sim J(r) \quad \text{for all} \quad k \in [a, b].$$

This proves (i). For the proof of (ii) consider $\phi(r) = J(r)(\log r)^\alpha$, then $\phi'(r) > 0$ for $r \geqslant r_0$. And so $\phi(r)$ is strictly increasing for $r \geqslant r_0$. Next let $\varepsilon > 0$ be such that $\varepsilon < \alpha$. Then

$$r \log r \frac{J'(r)}{J(r)} > -\varepsilon \quad \text{for all} \quad r \geqslant r_0 \quad \text{and so} \quad \int_{r_0}^{r} \frac{J'(t)}{J(t)} \mathrm{d}t > -\varepsilon \int_{r_0}^{r} \frac{1}{t \log t} \mathrm{d}t.$$

And so

$$\frac{J(r)}{J(r_0)} > \left(\frac{\log r_0}{\log r}\right)^\varepsilon.$$

Hence

$$(\log r)^\alpha J(r) > J(r_0)(\log r_0)^\varepsilon (\log r)^{\alpha - \varepsilon} \to \infty \quad \text{as} \quad r \to \infty$$

since $\alpha - \varepsilon > 0$. This proves (ii). The proof of (iii) follows as in (ii) by considering

$$\frac{r \log r J'(r)}{J(r)} < \varepsilon \quad \text{for all } r \geqslant r_0$$

and the fact that $[(\log r)^n / r] \to 0$ as $r \to \infty$ for any positive number n. To prove (iv) we start with integrating by parts to obtain

$$\int_{r_0}^{r} \left[\frac{(\log t)^\alpha J(t)}{t} \mathrm{d}t = J(t) \frac{(\log t)^{\alpha+1}}{\alpha + 1}\right]_{r_0}^{r} - \int_{r_0}^{r} \frac{(\log t)^{\alpha+1} J'(t)}{\alpha + 1} \mathrm{d}t. \tag{3}$$

Now

$$\int_{r_0}^{r} \frac{(\log t)^{\alpha+1} J'(t)}{\alpha + 1} \mathrm{d}t < \int_{r_0}^{r} \frac{\varepsilon J(t)(\log t)^\alpha}{t(\alpha + 1)} \mathrm{d}t = o\left(\int_{r_0}^{r} \frac{J(t)(\log t)^\alpha}{t} \mathrm{d}t\right).$$

From (3) it now follows that

$$(1 + o(1)) \int_{r_0}^{r} \frac{(\log t)^\alpha J(t)}{t} \mathrm{d}t = \frac{J(r)}{\alpha + 1} \{(\log r)^{\alpha+1} + o(1)\}.$$

And so

$$\int_{r_0}^{r} \frac{(\log r)^\alpha J(t)}{t} \mathrm{d}t \sim \frac{J(r)(\log r)^{\alpha+1}}{\alpha + 1}.$$

We finally prove (v). Let

$$I(r) = \int_{r}^{\infty} \frac{J(t)}{t(\log t)^\beta} \mathrm{d}t.$$

Once again integrating by parts, we have

$$I(r) = J(t) \frac{(\log t)^{-\beta+1}}{-\beta + 1}\Big]_{r}^{\infty} - \int_{r}^{\infty} \frac{(\log t)^{-\beta+1}}{-\beta + 1} J'(t) \mathrm{d}t$$

$$= \frac{J(r)}{(\beta - 1)(\log r)^{\beta-1}} + \frac{1}{\beta - 1} \int_{r}^{\infty} \frac{J'(t)}{(\log t)^{\beta-1}} \mathrm{d}t. \tag{4}$$

Since $J(r)$ is very slowly changing function, it follows that

$$\frac{J'(r)}{(\log r)^{\beta-1}} < \frac{J(r)}{r(\log r)^{\beta}} \quad \text{for all} \quad r \geqslant r_0.$$

Hence

$$\int_r^{\infty} \frac{J'(t)}{(\log t)^{\beta-1}} \mathrm{d}t < \varepsilon \int_r^{\infty} \frac{J(t)}{t(\log t)^{\beta}} \mathrm{d}t = o(I(r)).$$

The result (v) now follows from (4). This completes the proof of Theorem 1.

We next prove

Theorem 2. *If*

$$\int_{r_0}^{r} \frac{\phi(t)}{t} \mathrm{d}t \sim (\log r)^{\rho} J(r)$$

where $\phi(r)$ is monotonically increasing function, then $\phi(r) \sim \rho(\log r)^{\rho-1} J(r)$.

Proof. By hypothesis

$$(1-\varepsilon)(\log r)^{\rho} J(r) < \int_{r_0}^{r} \frac{\phi(t)}{t} \mathrm{d}t < (1+\varepsilon)(\log r)^{\rho} J(r).$$

Now if $k > 1$ then

$$\int_r^{r^k} \frac{\phi(t)}{t} \mathrm{d}t = \int_{r_0}^{r^k} \frac{\phi(t)}{t} \mathrm{d}t - \int_{r_0}^{r} \frac{\phi(t)}{t} \mathrm{d}t$$

$$< (1+\varepsilon)(\log r^k)^{\rho} J(r^k) - (1-\varepsilon)(\log r)^{\rho} J(r).$$

But by Theorem 1 (i), $J(r^k) \sim J(r)$ and so

$$\int_r^{r^k} \frac{\phi(t)}{t} \mathrm{d}t < J(r)(\log r)^{\rho}(k^{\rho}(1+\varepsilon) - (1-\varepsilon)) \tag{5}$$

Also for $k > 1$,

$$\int_r^{r^k} \frac{\phi(t)}{t} \mathrm{d}t \geqslant \phi(r) \int_r^{r^k} \frac{1}{t} \mathrm{d}t = (k-1)\phi(r) \log r. \tag{6}$$

From (5) and (6) it now follows that

$$\frac{\phi(r)}{(\log r)^{\rho-1} J(r)} < \frac{1}{k-1}(k^{\rho}(1+\varepsilon) - (1-\varepsilon)). \tag{7}$$

Put $k = 1 + \varepsilon_1$. Since $\varepsilon, \varepsilon_1$ are arbitrary, taking $\varepsilon_1 = \varepsilon^{1/2}$ it is easily seen that the right hand side of (7) tends to ρ. Thus

$$\limsup_{r \to \infty} \frac{\phi(r)}{(\log r)^{\rho-1} J(r)} \leqslant \rho. \tag{8}$$

Similarly

$$\liminf_{r\to\infty} \frac{\phi(r)}{(\log r)^{\rho-1}J(r)} \geqslant \rho. \tag{9}$$

Combining (8) and (9) we have the desired result.

We now give some applications of earlier results.

Theorem 3. *Let f be an entire function, $J(r)$ be a very slowly changing function, $k>0$. If*

$$\limsup_{r\to\infty} \frac{\log\log M(r,f)}{J(r)(\log r)^k} = \sigma \quad 0<\sigma<\infty$$

then

$$\limsup_{r\to\infty} \frac{rM(r,f')}{M(r,f)\log M(r,f)(\log r)^{k-1}J(r)} \geqslant k\sigma$$

where as usual $M(r,f) = \max_{|z|=r} |f(z)|$.

Remark. As a particular case, by putting $k=1$ and $J(r)\equiv 1$, we have the following corollary.

COROLLARY

If f is an entire function of order σ, $0<\sigma<\infty$ then

$$\limsup_{r\to\infty} \frac{rM(r,f')}{M(r,f)\log M(r,f)} \geqslant \sigma.$$

Proof of Theorem 3. Since $M(r,f') \geqslant M'(r,f)$ where the derivative exists, (see [3, 15]), it is sufficient to show that

$$\limsup_{r\to\infty} \frac{rM'(r,f)}{M(r,f)\log M(r,f)(\log r)^{k-1}J(r)} \geqslant k\sigma.$$

So, suppose

$$\limsup_{r\to\infty} \frac{rM'(r,f)}{M(r,f)\log M(r,f)(\log r)^{k-1}J(r)} = a,$$

where $a<k\sigma$. Thus for given $\varepsilon>0$, there exists $r_0\geqslant 0$, such that for all $r\geqslant r_0$

$$\frac{M'(r,f)}{M(r,f)\log M(r,f)} < \frac{(a+\varepsilon)J(r)(\log r)^{k-1}}{r}.$$

Integrating from r_0 to r we get

$$\log\log M(r,f) - \log\log M(r_0,f) < (a+\varepsilon)\int_{r_0}^{r} \frac{J(t)(\log t)^{k-1}}{t}\,dt.$$

Since $\log\log M(r_0, f) = o(\log\log M(r, f))$, using Theorem 1 (iv), it follows that

$$\log\log M(r, f) < \frac{(a+\varepsilon)J(r)(\log r)^k}{k}$$

and so

$$\limsup_{r\to\infty} \frac{\log\log M(r, f)}{J(r)(\log r)^k} \leqslant \frac{a}{k} < \sigma$$

contradicting the hypothesis. This proves the theorem.

Theorem 4. *Let $\rho \geqslant 1$ and let $f(z)$ be an entire function such that $\limsup_{r\to\infty}\{T(r, f)/r^\rho\} = \sigma$ where $\sigma > 0$, and $T(r, f)$ is the usual Nevanlinna characteristic function. Then $[n(r, x)/(\log r)^\rho J(r)] \to 0$ for atmost one value of x.*

Proof. Suppose for $x = x_1, x_2$ we have

$$\frac{n(r, x)}{(\log r)^\rho J(r)} \to 0$$

then

$$N(r, x) = A + \int_{r_0}^{r} \frac{n(t, x)}{t}\,dt$$

$$< A + \varepsilon \int_{r_0}^{r} \frac{(\log t)^\rho J(t)}{t}\,dt$$

$$\sim A + \frac{\varepsilon J(r)(\log r)^{\rho+1}}{\rho+1} \quad \text{using (iv) of Theorem 1.}$$

Thus

$$N(r, x) = o(J(r)(\log r)^{\rho+1}) \quad \text{for} \quad x = x_1, x_2. \tag{10}$$

Now f is entire, and hence

$$T(r, f) = N(r, x_1) + N(r, x_2) + O(\log r).$$

Using (10) it now follows that

$$T(r, f) = o(J(r)(\log r)^{\rho+1}).$$

The contradiction now follows on using the hypothesis and (iii) of Theorem 1. This completes the proof.

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Zeta functions of orders in quadratic fields

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Abstract. The zeta function of a lattice over an order in a semisimple Q-algebra introduced by L Solomon, is a generalization of Dedekind zeta function and has an Euler product in terms of local zeta functions. In this paper, the local zeta functions of localized orders in quadratic fields and hence the global ones are computed first by using Solomon's combinatorial method. Next the same formulae are obtained by considering orders in algebras (over p-adic fields) obtained by completing quadratic fields. In the complete case the methods of integration initiated by Bushnell and Reiner to study Solomon's zeta functions are used.

Keywords. Zeta functions; quadratic fields; Q-algebra.

1. Introduction

In this paper we determine Solomon's zeta functions for lattices in a quadratic number field over $\mathbb{Z}$-orders in the field. We begin by briefly discussing the concepts involved, and fix the notation. For detailed discussion see [9], [10]. For facts about orders we refer to [7].

Let A be a finite dimensional semisimple $\mathbb{Q}$-algebra and let Λ be an R-order in A where R is either $\mathbb{Z}$ or the localization $\mathbb{Z}_p$ of $\mathbb{Z}$ at a rational prime p. Let L be a (full) Λ-lattice in a finitely generated A-module V. Solomon's zeta function $\zeta_\Lambda(L; s)$ for the Λ-lattice L is defined as

$$\zeta_\Lambda(L; s) = \Sigma |L:N|^{-s}, \tag{1}$$

where the sum is over all Λ-sublattices N of L. Here $|L:N|$ denotes the index of N in L, and s is a complex variable. In the case $V = A$ where A is a number field and $L = \Lambda$ is the ring of integers in A, then Solomon's zeta function for L coincides with Dedekind's zeta function for A.

If Λ is a $\mathbb{Z}$-order in the $\mathbb{Q}$-algebra A, for a rational prime p, we form $\Lambda_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} \Lambda = \mathbb{Z}_p \Lambda$ and $L_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} L = \mathbb{Z}_p L$. Then Λ_p is a $\mathbb{Z}_p$-order in A and L_p is a Λ_p-lattice in V. So we may form local zeta function $\zeta_{\Lambda_p}(L_p; s)$ by using (1). It is shown in [10] that the Euler product formula

$$\zeta_\Lambda(L; s) = \Pi \zeta_{\Lambda_p}(L_p; s) \tag{2}$$

holds; here the product is over all rational primes p. Now fix a rational prime p, and let $\mathscr{P}$ be the partially ordered set of all Λ_p-lattices in V ordered by reverse inclusion.

$\mathscr{P}$ is locally finite and has a Mobius function μ. Fix an indeterminate t and fo pair of Λ_p-lattices $N \subseteq M$ write $[N:M] = t^n$ if the index $|M:N| = p^n$. It is k [7, p.228] that the number of Λ_p-isomorphism classes of Λ_p-lattices in V is choose a complete set of representatives $L_1, L_2, \ldots, L_k$ of Λ_p-isomorphism clas $\mathscr{P}$. Define two $k \times k$ matrices $\mathscr{A} = [a_{ij}]$ and $\mathscr{Z} = [Z_{ij}]$ as follows:

$$a_{ij} = \Sigma \mu(L_i, M)[M:L_i]$$
$$Z_{ij} = \Sigma [M:L_i],$$

where both the sums run through all Λ_p-sublattices M of L_i isomorphic to L_j. be shown that $\mathscr{A}$ depends only on the labelling of the classes and is independ the choice of representative in each class. Furthermore, Solomon [9] proved t

$$\mathscr{A}^{-1} = \mathscr{Z}.$$

Now denote by $Z_i(t) = Z_i$ the sum of the ith row of $\mathscr{Z}$; it is clear that

$$Z_i(p^{-s}) = \zeta_{\Lambda_p}(L_i; s).$$

Thus the local zeta functions are known once we compute $\mathscr{A}$ and invert it.

In the next section, we give a complete description of the matrix $\mathscr{A}$ for Λ_p-la in a quadratic field A where Λ_p is a $\mathbb{Z}_p$-order in A. As a consequence we obtain ex formulae for the corresponding local zeta functions and hence global formulae

In contrast to Solomon's algebraic and combinatorial method, Bushnell and H [2, 3] in a series of papers successfully adopted p-adic integration in computa involving Solomon's zeta functions. In § 3, we use their method to work out local zeta functions.

2. Solomon's matrix $\mathscr{A}$ and zeta functions for quadratic orders

We begin by collecting some facts we need about $\mathbb{Z}_p$-orders in an algebraic nu field A. Throughout this section p will be a fixed but arbitrary rational prime otherwise specified. For definitions and basic facts about order, refer to [7].

If O is the ring of algebraic integers in A, then $\mathbb{Z}_p O$ is the unique maximal $\mathbb{Z}_p$ in A, denoted by Λ_0. It is well known that Λ_0 is a principal ideal domain and $= \Pi P_i^{e_i}$ where P_i are distinct prime ideals of O, then $p\Lambda_0 = \Pi M_i^{e_i}$ where M_i a distinct prime ideals of Λ_0. Now let Λ be a $\mathbb{Z}_p$-order in A. Since $p\mathbb{Z}_p$ is the u maximal ideal in the local ring $\mathbb{Z}_p$, it follows that $p\Lambda \subseteq \operatorname{Rad}\Lambda$, where $\operatorname{Rad}\Lambda$ Jacobson radical of Λ. We can now prove

Lemma 1. Let L be a Λ-lattice in a vector space V over the number field A, wh is a $\mathbb{Z}_p$-order in A. Let μ denote the Mobius function of the poset of Λ-lattices Then $\mu(L, M) = 0$ unless $M \supseteq pL$.

Proof. By a result due to Ph. Hall [8; Cor. to Prop. 2, § 5] $\mu(L, M) = 0$ unless an intersection of maximal Λ-sublattices of L. Since $p\Lambda \subseteq \operatorname{Rad}\Lambda$, the Λ-sublatt of L is contained in the intersection of all maximal Λ-sublattices of L. The l follows.

Let Λ_1 and Λ_2 be two $\mathbb{Z}_p$-orders in A containing a $\mathbb{Z}_p$-order Λ in A such that Λ_1 and Λ_2 are isomorphic as Λ-lattices. If ϕ is the Λ-isomorphism, it can be extended uniquely to an A-module isomorphism of A (as $\mathbb{Q}\Lambda = \mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2 = A$). Such an isomorphism of A is given by the multiplication by a non-zero element of A. So $\Lambda_1 = \alpha\Lambda_2$ for some $\alpha \in A$. Comparing the 'order' we then see that $\Lambda_1 = \Lambda_2$. (By the order $O(L)$ of a Λ-lattice L in A we mean $O(L) = \{x \in A : xL \subseteq L\}$. $O(L)$ is actually a $\mathbb{Z}_p$-order in A containing Λ.) Thus we have

Lemma 2. Let Λ_1 and Λ_2 be $\mathbb{Z}_p$-orders in a number field A containing a $\mathbb{Z}_p$-order Λ in A. If Λ_1 and Λ_2 are isomorphic as Λ-lattices, then $\Lambda_1 = \Lambda_2$. □

From now onwards let A be a quadratic field. We choose $\Theta \in A$ such that $A = \mathbb{Q}(\Theta)$ and $O = \mathbb{Z} \oplus \mathbb{Z}\Theta$. Then

$$\Lambda_0 = \mathbb{Z}_p \oplus \mathbb{Z}_p\Theta$$

is the unique maximal $\mathbb{Z}_p$-order in A. For each integer $r \geqslant 0$, let

$$\Lambda_r = \mathbb{Z}_p \oplus \mathbb{Z}_p p^r \Theta \tag{6}$$

so that Λ_r is a $\mathbb{Z}_p$-order in A of index p^r in Λ_0. If Λ is any $\mathbb{Z}_p$-order of index p^r in Λ_0, then $p^r\Lambda_0 \subseteq \Lambda$ so that $\Lambda_r = \Lambda$. Thus we conclude that the collection of all $\mathbb{Z}_p$-orders in A forms a chain

$$\Lambda_0 \supset \Lambda_1 \supset \cdots \supset \Lambda_r \supset \cdots \tag{7}$$

where Λ_r as given in (6) is the unique $\mathbb{Z}_p$-order of index p^r in Λ_0.

Any $\mathbb{Z}$-lattice L in the quadratic field A is invertible. [1, p. 198]. Since $O(\mathbb{Z}_p L) = \mathbb{Z}_p(O(L))$ and $\mathbb{Z}_p L^{-1} = (\mathbb{Z}_p L)^{-1}$ [7, pp. 109, 192] it follows that any $\mathbb{Z}_p$-lattice in A is invertible. Consequently one has

Theorem 1. *Any $\mathbb{Z}_p$-lattice L in A is principal i.e. there is $\alpha \in L$ such that $L = O(L)\alpha$.* □

A proof can be found in [6, p. 98] where it is attributed to Ihara. It is a special case of more general results [4, Theorems 35.5, 35.14]. For us the importance of theorem lies in the following corollary,

COROLLARY 1

For a $\mathbb{Z}_p$-order Λ_r in the quadratic field A, two Λ_r-lattices in A are Λ_r-isomorphic if and only if their orders coincide. Thus for a fixed $r \geqslant 0$, $\{\Lambda_s; 0 \leqslant s \leqslant r\}$ is a complete set of representatives of Λ_r-isomorphism classes of Λ_r-lattices in A.

Proof. Because of Theorem 1, if two Λ_r-lattices have the same order, they must be isomorphic as Λ_r-lattices. Conversely, if they are isomorphic their orders will be isomorphic as Λ_r-lattices. Since these orders contain Λ_r, the proof is complete by lemma 2. □

Now fix a $\mathbb{Z}_p$-order Λ_r ($r \geqslant 0$). In view of the preceding results, in order to know the zeta functions of Λ_r-lattices in A, we need to compute only the zeta functions of

the orders Λ_i $(0 \leqslant i \leqslant r)$ considering them as Λ_r-lattices. Let $\mathscr{A}(r) = \mathscr{A}$ and $\mathscr{Z}(r) = \mathscr{Z}$ be the matrices associated to the order Λ_r as in (3). Since $\{\Lambda_0, \Lambda_1, \ldots, \Lambda_r\}$ is a full set of non-isomorphic Λ_r-lattices in the poset of Λ_r-lattices in A, both $\mathscr{A}(r)$ and $\mathscr{Z}(r)$ are $(r+1) \times (r+1)$ matrices. Let μ_r denote the Mobius function of this poset; then the entries of $\mathscr{A}(r)$ are given by

$$a_{ij}(r) = \Sigma \mu_r(\Lambda_i, M) \quad [M:\Lambda_i], \tag{8}$$

where the sum is over all Λ_r-sublattices M in Λ_i such that $p\Lambda_i \subseteq M$ and $M \cong \Lambda_j$ as Λ_r-lattices (see lemma 1). Also as mentioned in the introduction, $\mathscr{Z}(r) = \mathscr{A}(r)^{-1}$ and

$$\zeta_{\Lambda_r}(\Lambda_i; s) = Z_i(r), \tag{9}$$

where $Z_i(r)$ is the sum of the ith row of the matrix $\mathscr{Z}(r)$. Note that in both the matrices $\mathscr{A}(r)$ and $\mathscr{Z}(r)$, the subscripts i and j run from 0 to r. Our next result describes the nature of $\mathbb{Z}_p$-lattices between Λ_i and $p\Lambda_i$.

Theorem 2. *Let M be a $\mathbb{Z}_p$-lattice lying between Λ_i and $p\Lambda_i$; $M \neq \Lambda_i$, $M \neq p\Lambda_i$. $(i \geqslant 0)$. Let $\varepsilon(p) = -1, 0$ or 1 depending on whether p is inert, ramifies or decomposes in A.*

(i) *If $i = 0$, then the order $O(M)$ of M is either Λ_0 or Λ_1. The number of M with $O(M) = \Lambda_0$ is $\varepsilon(p) + 1$, and with $O(M) = \Lambda_1$ is $p - \varepsilon(p)$.*

(ii) *If $i > 0$, then $O(M)$ is either Λ_{i-1} or Λ_{i+1}. The lattice $p\Lambda_{i-1}$ is the only one having the order Λ_{i-1}. The number of M with $O(M) = \Lambda_{i+1}$ is exactly p.*

Proof. We first observe that if for such a lattice $M, O(M) = \Lambda_s$ then $s \leqslant i+1$, for otherwise $\Lambda_s \subset \Lambda_i$. Also by Theorem 2, we may choose $\alpha \in M$ such that $M = \Lambda_s\alpha$. Now multiplying $p\Lambda_i \subset M = \Lambda_s\alpha \subset \Lambda_i$ by Λ_i we obtain $p\Lambda_i \subset \Lambda_s\alpha \subset \Lambda_i\alpha = \Lambda_i$ which implies $p^{s-i} = p$ contradicting $s > i+1$. This observation proves the first assertion in (i). To prove the corresponding assertion in (ii), assume $O(M) = \Lambda_s$ where $s \leqslant i$. Now multiply $p\Lambda_i \subset M \subset \Lambda_i$ by Λ_s to obtain $p\Lambda_s \subseteq M \subset \Lambda_s$ whence

$$p^2 \geqslant |\Lambda_s:M| = |\Lambda_s:\Lambda_i| \quad |\Lambda_i:M| = p|\Lambda_s:\Lambda_i|.$$

Thus $|\Lambda_s:\Lambda_i| \leqslant p$ so that either $\Lambda_s = \Lambda_{i-1}$ or $\Lambda_s = \Lambda_i$. Our first assertion in (ii) is proved once we exclude the second possibility. So let $O(M) = \Lambda_i$; we aim to obtain a contradiction. As before, we write $M = \Lambda_i\alpha$ for some $\alpha \in M$. In that case, multiplying $p\Lambda_i \subset M = \Lambda_i\alpha \subset \Lambda_i$ by Λ_0 we obtain $p\Lambda_0 \subseteq \Lambda_0\alpha \subseteq \Lambda_0$.

If $\Lambda_0\alpha = \Lambda_0$, then $\alpha \in U(\Lambda_0)$, the group of units in Λ_0. But $\alpha \in \Lambda_i$ and $U(\Lambda_0) \cap \Lambda_i = U(\Lambda_i)$ [7, Ex 25.4, p. 224] so $M = \Lambda_i\alpha = \Lambda_i$, a contradiction. If $\Lambda_0\alpha = p\Lambda_0$, then $\alpha \in p\Lambda_0 \cap \Lambda_i = p\Lambda_{i-1}$ and thus $M \subseteq p\Lambda_{i-1} \subset \Lambda_i$. As $|\Lambda_i:M| = p$, these inclusions imply that $M = p\Lambda_{i-1}$ which contradicts $O(M) = \Lambda_i$. Thus we can assume that $|\Lambda_0:\Lambda_0\alpha| = p$. In that case $\Lambda_0\alpha \cap \Lambda_1 = p\Lambda_0$ so $\alpha \in p\Lambda_0$ and $\Lambda_0\alpha \subseteq p\Lambda_0$, a contradiction. These show that $O(M) = \Lambda_s \neq \Lambda_i$.

To obtain the other assertions, note that if $s > i$, then $\Lambda_i/p\Lambda_i$ is a two-dimensional vector space over the field $\Lambda_s/p\Lambda_{s-1}$ of p elements and that the number of Λ_s-sublattices of Λ_i containing $p\Lambda_i$ and having index p in Λ_i is exactly $(p+1)$. On the other hand it is clear that the number of sublattices of Λ_0 having index p in Λ_0 with their order as Λ_0 is $\varepsilon(p) + 1$. Hence the second assertion in (i). If $i > 0$, we have already shown that the $\mathbb{Z}_p$-sublattices of index p in Λ_i are all Λ_{i+1}-sublattices so they number $(p+1)$ by the preceding observation. If, for such a lattice M, $O(M) = \Lambda_{i-1}$, then the inclusions

$p\Lambda_i \subset M \subset \Lambda_i$ yield $p\Lambda_{i-1} \subseteq M \subset \Lambda_i$ which forces $M = p\Lambda_{i-1}$. This completes the proof of (ii). □

Note that a consequence of the last result is the following

COROLLARY 2

For $i > 0$, the $\mathbb{Z}_p$-order Λ_i is a local ring with $p\Lambda_{i-1}$ as the unique maximal ideal. □

Now that we know the number and the isomorphism classes of the required lattices, we just have to compute the values of the Mobius function to complete the description of $\mathscr{A}(r)$. Using the information of the preceding result and the recursive definition of μ_r we can compute these values easily (see [9] and [10]; for the sake of completeness we give the recursive definition of the Mobius function μ of a poset $\mathscr{P}$:

$$\mu(L, L) = 1, \quad \Sigma\mu(L, N) = 0 \quad \text{if} \quad L \supseteq M, \quad L \neq M,$$

where the sum is over all N in $\mathscr{P}$, such that

$$L \supseteq N \supseteq M$$

$$\mu(L, M) = 0 \quad \text{otherwise.})$$

We list these values

$$\mu_0(\Lambda_0, p\Lambda_0) = \varepsilon(p),$$

$$\mu_r(\Lambda_r, p\Lambda_r) = 0 \qquad \text{if} \quad r \geqslant 1,$$

$$\mu_r(\mu_i, p\Lambda_i) = p \qquad \text{if} \quad 0 \leqslant i \leqslant r,$$

$$\mu_r(\Lambda_i, M) = -1 \qquad \text{if} \quad 0 \leqslant i \leqslant r, \quad M \ \Lambda_r\text{-sublattice of } \Lambda_i \text{ of index } p.$$

Putting all these in (8), we can easily obtain the following description of the matrix $\mathscr{A}(r)$.

Theorem 3. *Let r be a non-negative integer and let $\mathscr{A}(r) = [a_{ij}]$ be the $(r+1) \times (r+1)$ matrix as defined in* (8).

If $r = 0$, then

$$a_{00} = 1 - (\varepsilon(p) + 1)t + \varepsilon(p)t^2.$$

If $r \geqslant 1$, then $a_{ij} = 0$ unless $|i - j| \leqslant 1$. The remaining entries are

$$a_{00} = 1 - (\varepsilon(p) + 1)t + pt^2; \quad a_{01} = -(p - \varepsilon(p))t,$$

$$a_{rr} = 1,$$

$$a_{ii} = 1 + pt^2; \quad 0 < i < r,$$

$$a_{i,i+1} = -pt; \quad 0 < i < r,$$

$$a_{i+1,i} = -t; \quad 0 \leqslant i < r.$$

The picture of $\mathscr{A}(r)$ for $r > 1$ obtained in Theorem 3 makes it clear that the suitable row and column operations allow us to link $\mathscr{A}(r)$ to $\mathscr{A}(r-1)$. To be precise, let E_{ij} denote the elementary $(r+1) \times (r+1)$ matrix with 1 at (i, j)th place and zeros everywhere else; and let I be the $(r+1) \times (r+1)$ identity matrix. Then using the picture

of $\mathscr{A}(r)$ in Theorem 3 one easily verifies that

$$(I + ptE_{r-1,r})\mathscr{A}(r)(I + tE_{r,r-1}) = \left[\begin{array}{c|c} \mathscr{A}(r-1) & 0 \\ \hline 0 & 1 \end{array}\right], \tag{10}$$

$$(I + (p - \varepsilon(p))tE_{0,1})\mathscr{A}(1)(I + tE_{1,0}) = \left[\begin{array}{c|c} \mathscr{A}(0) & 0 \\ \hline 0 & 1 \end{array}\right]. \tag{11}$$

For simplicity, fix the following notation

$$\phi(t) = (1-t)(1-\varepsilon(p)t), \tag{12}$$

$$h(p) = p - \varepsilon(p).$$

The preceding relations then yield the following interesting result:

COROLLARY 3

For $r \geqslant 0$,

$$\det \mathscr{A}(r) = \phi(t),$$
$$\det \mathscr{Z}(r) = \phi(t)^{-1}. \qquad \square$$

Note that $\phi(t)^{-1} = Z_0(0)$ *is the local zeta function of the maximal order* Λ_0.

Now inverting the matrix

$$\mathscr{A}(1) = \begin{bmatrix} 1 - (\varepsilon(p)+1)t + pt^2 & -h(p)t \\ -t & 1 \end{bmatrix}$$

where we use the fact that $\det \mathscr{A}(1) = \phi(t)$, we obtain

$$Z_0(1) = \phi(t)^{-1}[1 + h(p)t], \tag{13}$$

$$Z_1(1) = \phi(t)^{-1}[1 - \varepsilon(p)t + pt^2], \tag{14}$$

they being the sum of the first and the second rows of $\mathscr{Z}(1)$. For future reference we list two individual entries of $\mathscr{Z}(1)$ also:

$$Z_{0,1}(1) = \phi(t)^{-1}h(p)t, \tag{15}$$

$$Z_{1,1}(1) = \phi(t)^{-1}[1 - (1 + \varepsilon(p))t + pt^2].$$

To obtain the corresponding "partial" zeta functions for $r \geqslant 2$, we first take inverse in the matrix equation (10). This yields

$$\mathscr{Z}(r) = (I + tE_{r,r-1})\left[\begin{array}{c|c} \mathscr{Z}(r-1) & 0 \\ \hline 0 & 1 \end{array}\right](I + ptE_{r-1,r}).$$

Introducing the matrix

$$X = [x_{ij}] = \left[\begin{array}{c|c} Z(r-1) & 0 \\ \hline 0 & 1 \end{array}\right]$$

ıe last equation may be rewritten as

$$\mathscr{X}(r) = X + t\sum_{j=0}^{r} x_{r-1,j}E_{rj} + pt\sum_{i=0}^{r} X_{i,r-1}E_{ir} + pt^2 x_{r-1,r-1}E_{rr}.$$

Iow, for $0 \leqslant i < r$, the sum of the ith row of X is $Z_i(r-1)$ whereas the sum of the ıst row i.e. the rth row is just 1. Therefore one can easily deduce the following from ıe preceding matrix equation:

$$\begin{aligned}
&\text{(i)}\quad Z_i(r) = Z_i(r-1) + ptZ_{i,r-1}(r-1) && 0 \leqslant i < r,\\
&\text{(ii)}\quad Z_{ir}(r) = ptZ_{i,r-1}(r-1) && 0 < i < r,\\
&\text{(iii)}\quad Z_{rr}(r) = 1 + pt^2 Z_{r-1,r-1}(r-1)\\
&\text{(iv)}\quad Z_r(r) = tZ_{r-1}(r-1) + Z_{rr}(r).
\end{aligned} \tag{16}$$

The main result shows that $\zeta_{\Lambda_r}(\Lambda_i; s)$ is a product of $\phi(p^{-s})^{-1}$ and certain olynomials in p^{-s}. We first define these polynomials: For any integer k, let the symbol $k; t]$ denote the rational function in t given by

$$[k; t] = (1 - t^k)/(1 - t)$$

o that for $k \geqslant 1$, $[k; t]$ is a polynomial in t whereas $[0; t] = 0$. Now for integers i and , $0 \leqslant i \leqslant r$, and a rational prime p let

$$f_p(r, i; t) = [r - i + 1; pt] + (pt)^{r-i+1} \cdot t[i - 1; pt^2].$$

t is clear that $f_p(r, i; t)$ is a polynomial in t if $1 \leqslant i \leqslant r$. One readily verifies that for ntegers i, r such that $1 \leqslant i \leqslant r$, we have

$$f_p(r, i; t) - tf_p(r, i-1; t) = (1 - t)[r - i + 1; pt]. \tag{17}$$

Theorem 4. *Let p be a rational prime and let $\phi(t) = (1 - t)(1 - \varepsilon(p)t)$. Let i, r be integers such that $0 \leqslant i \leqslant r$, and let Λ_i and Λ_r be $\mathbb{Z}_p$-orders in the quadratic field A having index p^i and p^r respectively in the maximal $\mathbb{Z}_p$-order Λ_0 in A. Then*

$$\zeta_{\Lambda_r}(\Lambda_i; s) = \phi(p^{-s})^{-1}[(1 - \varepsilon(p)p^{-s})f_p(r, i; p^{-s}) + p^r p^{-s(r+i)}].$$

Proof. It suffices to prove that

$$Z_i(r) = \phi(t)^{-1}[(1 - \varepsilon(p)t)f_p(r, i; t) + p^r t^{r+i}]. \tag{18}$$

Case I: $i = 0$. In this case if $r = 0$, then (18) is $Z_0(0) = \phi(t)^{-1}$ which we have verified already. So we assume $r > 0$. Writing out the right hand side of (18) in terms of $h(p)$ where $h(p) = p - \varepsilon(p)$, we can put (18) in the form

$$Z_0(r) = \phi(t)^{-1}\left[1 + h(p)t\sum_{j=0}^{r-1}(pt)^j\right]. \tag{19}$$

This we prove by induction on r. If $r = 1$, (19) is just the relation (13). So assume $r > 1$. In this case note that the second relation in (16) together with (15) implies

$$Z_{0,r-1}(r-1) = \phi(t)^{-1}h(p)t(pt)^{r-2}.$$

Using this in the first relation of (16) and applying the induction hypothesis we can obtain (19) easily.

Case II: $i > 0$. We first note that the following relation

$$Z_i(r) = Z_i(i) + pt[r-1; pt]Z_{ii}(i); \quad 1 \leqslant i \leqslant r \tag{20}$$

can be obtained from (i) and (ii) of (16). Now coming back to the proof of (18) in the present case of $i > 0$, we use induction on i. The case of $i = 1$ is a straightforward substitution of values of $Z_1(1)$ and $Z_{1,1}(1)$ from (14) and (15) in (20). So we may assume $i > 1$. First we put (20) in the form

$$Z_i(r) = tZ_{i-1}(i-1) + [r-i+1; pt]\{1 + pt^2 Z_{i-1,i-1}(i-1)\}$$

using the values of $Z_i(i)$ and $Z_{ii}(i)$ from (16).

On the other hand, since $i > 1$, we may replace $i-1$ by i in (20) to obtain

$$Z_{i-1}(r) = Z_{i-1}(i-1) + pt[r-i+1; pt]Z_{i-1,i-1}(i-1).$$

These two relations imply that

$$Z_i(r) = tZ_{i-1}(r) + [r-i+1; pt],$$

whence (18) follows by the induction hypothesis and (17). □

In the special case of $i = r$, since $f_p(r, r; t) = [r; pt^2]$ a simple calculation shows that

$$(1 - \varepsilon(p)t)f_p(r, r; t) + p^r t^{2r} = \frac{(1 - \varepsilon(p)t)(1 - (pt^2)^r) + (1 - pt^2)(pt^2)^r}{1 - pt^2}.$$

Thus, in the special case of $i = r$, the formula for the corresponding zeta function in the preceding theorem reduces to the one given by Galkin [5]. Galkin's formula is for odd primes only, and is obtained by solving congruences modulo powers of primes.

The deduction of the formulae for the global case is now fairly routine. Let Γ be the maximal $\mathbb{Z}$-order in the quadratic field A, Λ an arbitrary $\mathbb{Z}$-order and L a Λ-lattice in A; the subscript p as before denotes the localization at a rational prime p. We are interested in an explicit formula for $\zeta_\Lambda(L; s)$. Now Γ_p being the maximal $\mathbb{Z}_p$-order in A, if $\zeta_A(s)$ denotes the Dedekind zeta function for the quadratic field A, then

$$\zeta_A(s) = \Pi \zeta_{\Gamma_p}(\Gamma_p; s)$$

by (2), where the product is over all rational primes p. Since theorem 4 shows that

$$\zeta_{\Gamma_p}(\Gamma_p; s) = \phi_p(p^{-s})^{-1}$$

for any rational prime p (where $\phi_p(t)$ is the polynomial $(1-t)(1-\varepsilon(p)t)$ it follows that

$$\zeta_A(s) = \Pi \phi_p(p^{-s})^{-1}. \tag{21}$$

The product being over all rational primes p. Now given a $\mathbb{Z}$-order Λ in A, there is a set $E(\Lambda)$ of exceptional primes p, (the set of primes that divide the index $|\Gamma : \Lambda|$) such that $\Gamma_p = \Lambda_p$ if and only if $p \notin E(\Lambda)$. Consequently for primes $p \notin E(\Lambda)$, $\zeta_{\Lambda_p}(L_p; s) = \zeta_{\Gamma_p}(\Gamma_p; s)$. On the other hand, if $p \in E(\Lambda)$ and if the $\mathbb{Z}_p$-order isomorphic to L_p has

index p^i in Γ_p, then our last theorem implies that

$$\zeta_{\Lambda_p}(L_p; s)/\zeta_{\Gamma_p}(\Gamma_p; s) = (1 - \varepsilon(p)p^{-s}) f_p(r, i; p^{-s}) + p^{r(1-s)-si},$$

where r is determined by the relation $p^r = |\Gamma_p : \Lambda_p|$. $(r > 0)$. The right hand side of the preceding equation is a polynomial in p^{-s}; denote it by $g_p(p^{-s})$. Therefore, the Euler product expression for $\zeta_\Lambda(L; S)$ together with (21) yield the following global formula:

$$\zeta_\Lambda(L; s) = \zeta_A(s) \Pi g_p(p^{-s}), \tag{22}$$

where the product is over all the primes p in the finite set $E(\Lambda)$.

3. Computations using the method of Bushnell and Reiner

Shortly after the appearance of Solomon's [9, 10] papers, Bushnell and Reiner [2, 3] gave an alternative treatment of Solomon's zeta functions by adopting the idea of "zeta integrals on the idele group". Their treatment requires working with lattices over orders in semisimple algebras over the p-adic field $\hat{\mathbb{Q}}_p$, so that suitable topology can be introduced and a Haar measure can be chosen on certain entities related to these algebras. These lead to the formulation of the zeta functions in terms of Haar integrals. We do not go into the details, for our aim is limited to use these integrals in our special case of orders in quadratic fields. In our case, this new machinery may not appear to simplify the matters, but there can be little doubt that it is more powerful, at least for theoretical purposes, than the combinatorial methods of Solomon. For facts about orders with respect to complete base fields, we refer to Reiner [7].

Let $\hat{\mathbb{Q}}_p$ and $\hat{\mathbb{Z}}_p$ denote the p-adic field and its ring of integers ($\mathbb{Z}_p$ still denotes the localization of $\mathbb{Z}$ at p). If Λ is a $\mathbb{Z}_p$-order in a finite dimensional semisimple $\mathbb{Q}$-algebra A, and L a Λ-lattice in a finitely generated A-module V, then by setting

$$\hat{A} = \hat{\mathbb{Q}}_p \underset{\mathbb{Q}}{\otimes} A, \quad \hat{\Lambda} = \hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}_p} \Lambda, \quad \hat{V} = \hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} V, \quad \hat{L} = \hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}_p} L.$$

We see that $\hat{\Lambda}$ is a $\hat{\mathbb{Z}}_p$-order in $\hat{A}$, $\hat{V}$ a finitely generated $\hat{A}$-module and $\hat{L}$ a $\hat{\Lambda}$-lattice in $\hat{V}$. Now, in a manner completely analogous to the definition (1), we can define the zeta function $\zeta_{\hat{\Lambda}}(\hat{L}; s)$. The natural map between Λ-lattices in V and $\hat{\Lambda}$-lattices in $\hat{V}$ allow us to conclude that

$$\zeta_\Lambda(L; s) = \zeta_{\hat{\Lambda}}(\hat{L}; s). \tag{23}$$

See [lemma 9; 10]. This relation shows that even for global formulae, we may as well work with completions.

As in earlier sections, let $A = \mathbb{Q} \dot{\oplus} \mathbb{Q}\Theta$ be a quadratic field. Since we shall be working with a fixed prime p, we shall drop the subscript p henceforth while discussing p-adic field or its integers. Then $\hat{A} = \hat{\mathbb{Q}} \oplus \hat{\mathbb{Q}}\Theta$. We note that, depending on the nature of the minimal polynomial of a primitive element of A over $\mathbb{Q}$, $\hat{A}$ is either a quadratic extension of $\hat{\mathbb{Q}}$ or $\hat{A}$ is isomorphic to the sum of two copies of $\hat{\mathbb{Q}}$ as a $\hat{\mathbb{Q}}$-algebra. [§ 5c; 7]. We set

$$\hat{\Lambda}_r = \hat{\mathbb{Z}} \oplus p^r \hat{\mathbb{Z}}\theta, \quad r = 0, 1, 2, \ldots.$$

Then from what we have already seen about $\mathbb{Z}_p$-orders in a quadratic field, we conclude that $\hat{\Lambda}_0$ is the maximal $\hat{\mathbb{Z}}$-order in $\hat{A}$ and that the chain

$$\hat{\Lambda}_0 \supset \hat{\Lambda}_1 \supset \hat{\Lambda}_2 \supset \cdots$$

gives all the $\hat{\mathbb{Z}}$-orders in $\hat{A}$. The result about isomorphism classes too carries over. So for a fixed $r \geqslant 0$, the isomorphism classes of $\hat{\Lambda}_r$-lattices in A are represented by the $\hat{\Lambda}_r$-lattices $\{\hat{\Lambda}_i : 0 \leqslant i \leqslant r\}$.

We now set $V = \hat{A}$ itself so that we can discuss $\zeta_{\hat{\Lambda}_i}(\hat{\Lambda}_r; s)$ in its proper setting. We define the partial zeta functions $\hat{Z}_{ij}$ as follows

$$\hat{Z}_{ij} = \Sigma |\hat{\Lambda}_i : M|^{-s},$$

where s is a complex variable and the sum is over all $\hat{\mathbb{Z}}$-sublattices M of $\hat{\Lambda}_i$ such that $M \cong \hat{\Lambda}_j$. Thus for fixed positive integers i and r $(0 \leqslant i \leqslant r)$, we see that the zeta function

$$\zeta_{\hat{\Lambda}_r}(\hat{\Lambda}_i, s) = \sum_{j=0}^{r} \hat{Z}_{ij}. \tag{24}$$

Thus as in § 2, we have to evaluate the functions $\hat{Z}_{ij}$. Note that $\hat{Z}_{0,0}$ is nothing but the zeta function of the maximal order $\hat{\Lambda}_0$. So we know this explicitly. If $\hat{A}$ remains a quadratic extension of $\hat{\mathbb{Q}}$, $\hat{Z}_{0,0}$ depends on the ramification of p in $\hat{A}$ and will be the same as in the formula found in § 2. If $\hat{A} \simeq \hat{\mathbb{Q}} \times \hat{\mathbb{Q}}$, then

$$\hat{Z}_{0,0} = (1 - p^{-s})^{-2}.$$

To evaluate $\hat{Z}_{ij}$ in general, we first express it as an integral following Bushnell and Reiner [2, § 3.4]. Our task is simplified here, for we are taking $V = \hat{A}$. Now $\hat{A}$ as a finite dimensional $\hat{\mathbb{Q}}$-algebra, inherits a topology which makes it a locally compact topological algebra; the units $\hat{A}^X$ of $\hat{A}$ form a locally compact topological group. Let a suitably normalized Haar measure $d^X x$ be chosen on $\hat{A}^X$, and let $\mu^X(S)$ denote the measure of a subset S of $\hat{A}^X$. For $x \in \hat{A}^X$, let $\|x\|$ denote its norm defined in terms of the 'generalized' index as in [2]; namely $\|x\|$ is the 'generalized' index $|Mx : M|$ for any $\hat{\mathbb{Z}}$-lattice M in $\hat{A}$ where

$$|Mx : M| = Mx : Mx \cap M| / |M : Mx \cap M|.$$

The norm $\|x\|$ independent of the lattice M, is multiplicative, and if x is a unit in some $\hat{\mathbb{Z}}$-order in $\hat{A}$, then $\|x\| = 1$. Now adopting formula (11) of [2] to our case, we obtain

$$\hat{Z}_{ij} = \mu^X(\hat{\Lambda}_j^X)^{-1} |\hat{\Lambda}_i : \hat{\Lambda}_j|^{-s} \int_{\hat{A}^X \cap \{\hat{\Lambda}_j : \hat{\Lambda}_i\}} \|x\|^s \, d^X x, \quad 0 \leqslant j \leqslant i, \tag{25}$$

where

$$\begin{aligned} \{\hat{\Lambda}_j : \hat{\Lambda}_i\} &= \{x \in \hat{A} : \hat{\Lambda}_j x \subseteq \hat{\Lambda}_i\} \\ &= \{x \in \hat{\Lambda}_i : \hat{\Lambda}_j x \subseteq \hat{\Lambda}_i\} \\ &= p^{i-j} \hat{\Lambda}_j. \end{aligned}$$

We affect a change of variables in the integral of (25) by letting $x = p^{i-j} y$, $y \in \hat{\Lambda}_j$. Then

$$d^X x = d^X y, \quad \|x\| = (p^{-2})^{i-j} \|y\|$$

so

$$\int_{\hat{A}^X\cap\{\hat{\Lambda}_j:\hat{\Lambda}_i\}} \|x\|^s \mathrm{d}^X x = p^{-2s(i-j)} \int_{\hat{A}^X\cap\hat{\Lambda}_j} \|y\|^s \mathrm{d}^X y.$$

Also note that $|\hat{\Lambda}_i:\hat{\Lambda}_j|^{-1} = p^{i-j}$. Thus we have the following form of (25)

$$\hat{Z}_{ij} = \mu^X(\hat{\Lambda}_j^X)^{-1} p^{-s(i-j)} \int_{\hat{A}^X\cap\hat{\Lambda}_j} \|x\|^s \mathrm{d}^X x. \tag{26}$$

One easily concludes that

$$\hat{Z}_{ij} = p^{-s(i-j)} \hat{Z}_{jj}, \quad 0 \leqslant j \leqslant i. \tag{27}$$

This is precisely our formula (ii) of (16). It looks better as there is no extraneous matrix to keep track of.

We now look for a recursive formula for $\hat{Z}_{jj}$, which according to (26) has the form

$$\hat{Z}_{jj} = \mu^X(\hat{\Lambda}_j^X)^{-1} \int_{\hat{A}^X\cap\hat{\Lambda}_j} \|x\|^s \mathrm{d}^X x, \quad j \geqslant 0. \tag{28}$$

For $j=0$, we already know $\hat{Z}_{00}$, for it is the zeta function of the maximal order $\hat{\Lambda}_0$. So we assume $j \geqslant 1$. Now to handle the integral in (28), we break up the range of integration $\hat{A}^X \cap \hat{\Lambda}_j$ into two non-overlapping subsets, $\hat{A}^X \cap \operatorname{Rad} \hat{\Lambda}_j$ and $\hat{A}^X \cap (\hat{\Lambda}_j - \operatorname{Rad} \hat{\Lambda}_j)$. Note that for $j \geqslant 1$, $\hat{\Lambda}_j$ is a local ring; in fact

$$\hat{\Lambda}_j = \hat{\mathbb{Z}} \oplus p^j \mathbb{Z}\theta, \quad \operatorname{Rad} \hat{\Lambda}_j = p\hat{\mathbb{Z}} \oplus p^j \hat{\mathbb{Z}}\theta. \tag{29}$$

Since $\operatorname{Rad} \hat{\Lambda}_j = p\hat{\Lambda}_{j-1} (j \geqslant 1)$, it follows that

$$\begin{aligned} \int_{\hat{A}^X\cap\operatorname{Rad}\hat{\Lambda}_j} \|x\|^s \mathrm{d}^X x &= p^{-2s} \int_{\hat{A}^X\cap\hat{\Lambda}_{j-1}} \|y\|^s \mathrm{d}^X y \\ &= p^{-2s} \mu^X(\hat{\Lambda}_{j-1}^X) \hat{Z}_{j-1,j-1}. \end{aligned} \tag{30}$$

To evaluate the integral over the other set, we note from (29) $\hat{\Lambda}_j/\operatorname{Rad} \hat{\Lambda}_j \cong \mathbb{Z}/p\mathbb{Z}$, whence comparing the unit groups of these rings (and using the fact that $\hat{\Lambda}_j$ is local) we obtain

$$\hat{\Lambda}_j^X/(1 + \operatorname{Rad} \hat{\Lambda}_j) \cong (\mathbb{Z}/p\mathbb{Z})^X$$

so that

$$\mu^X(\hat{\Lambda}_j^X) = (p-1)\mu^X(1 + \operatorname{Rad} \hat{\Lambda}_j).$$

Also for any $x \in \Lambda_j$ such that $x \notin \operatorname{Rad} \hat{\Lambda}_j$, x is a unit in $\hat{\Lambda}_j$ as well as in $\hat{A}$. Thus

$$\begin{aligned} \int_{\hat{A}^X\cap(\hat{\Lambda}_j - \operatorname{Rad}\hat{\Lambda}_j)} \|x\|^s \mathrm{d}^X x &= (p-1)\mu^X(1 + \operatorname{Rad} \hat{\Lambda}_j) \\ &= \mu^X(\hat{\Lambda}_j^X). \end{aligned} \tag{31}$$

Therefore, combining (30) and (31), and using (28), we obtain

$$\begin{aligned} \hat{Z}_{jj} &= 1 + p^{-2s}\{\mu^X(\hat{\Lambda}_{j-1}^X)/\mu^X(\hat{\Lambda}_j^X)\} \hat{Z}_{j-1,j-1} \\ &= 1 + p^{-2s}\{|\hat{\Lambda}_{j-1}^X : \hat{\Lambda}_j^X|\} \hat{Z}_{j-1,j-1}. \end{aligned}$$

Finally, we show how to compute $|\hat{\Lambda}_{j-1}^{X}:\hat{\Lambda}_{j}^{X}|$. Note that $\hat{\Lambda}_{j-1}^{X}$ acts transitively on the set of all $\hat{\mathbb{Z}}$-sublattices between $\hat{\Lambda}_{j-1}$ and $p\hat{\Lambda}_{j-1}$ and having their order (or coefficient ring) as $\hat{\Lambda}_{j}$. Since the stabilizer of any such sublattice is $\hat{\Lambda}_{j}^{X}$, we see that the index $|\hat{\Lambda}_{j-1}^{X}:\hat{\Lambda}_{j}^{X}|$ is precisely the number of sublattices between $\hat{\Lambda}_{j-1}$ and $p\hat{\Lambda}_{j-1}$ having $\hat{\Lambda}_{j}$ as their order. From Theorem 2, we conclude that

$$\begin{aligned} |\hat{\Lambda}_{j-1}^{X}:\hat{\Lambda}_{j}^{X}| &= p, \qquad j>1 \\ &= p-\varepsilon(p), \quad j=1. \end{aligned}$$

Thus the final form of the recursive relations for $\hat{Z}_{jj}$ is as follows:

$$\begin{aligned} \hat{Z}_{jj} &= 1+p^{-2s-1}\hat{Z}_{j-1,j-1}, \quad j>1 \\ \hat{Z}_{1,1} &= 1+p^{-2s}(p-\varepsilon(p))\hat{Z}_{00}. \end{aligned} \tag{32}$$

Note that in §2, in relations (iii) of (16) and (15) we had found these relations by Solomon's determinant method. It is now a routine work to compute various $\hat{Z}_{ij}$ (by using (27)) and then Solomon's zeta functions (using (25) and (23)).

Now that the two methods of calculating zeta functions of quadratic orders (one by evaluating determinants and the other by integration) may be compared, it may appear that the method of integration is superior due to its simplicity. But here it should be pointed out that Solomon's papers raise some other questions in addition to the straightforward calculation of zeta functions. See for example his conjecture about the determinant $\mathscr{A}(t)$ [9]. It seems certain that Solomon's combinatorial method will continue to be useful in questions relating to the determinant $\mathscr{A}(t)$.

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Absolute matrix summability of the allied series of a Fourier series

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Abstract. In this paper we have proved a theorem for infinite triangular matrix using new techniques.

Keywords. Absolute matrix summability; bounded variation; infinite triangular matrix.

1. Introduction

Let Σu_n be an infinite series with the sequence of its partial sums $\{s_n\}$. If $M = (a_{n,k})$ be an infinite matrix with real or complex elements, then the transform $\{t_n\}$ of $\{s_n\}$ is given by the matrix multiplication,

$$t_n = \sum_{k=0}^{\infty} a_{n,k} s_k, \tag{1}$$

(assuming that t_n exists for every $n \geqslant 0$). Now, if

$$\lim_{n\to\infty} t_n = s,$$

then the sequence $\{s_n\}$, or the series Σu_n is said to be summable $(a_{n,k})$ or simply M-summable to s. Also the series Σu_n is said to be absolutely summable by M-method, or simply summable $|M|$, if the corresponding auxiliary sequence $\{t_n\}$ is of bounded variation (written hereafter as b.v.), that is to say,

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty. \tag{2}$$

The matrix $M = (a_{n,k})$ is said to be regular if and only if

$$\lim_{n\to\infty} s_n = s \Rightarrow \lim_{n\to\infty} t_n = s$$

and absolutely regular if and only if $\{s_n\} \in$ b.v. $\Rightarrow \{t_n\} \in$ b.v.
The necessary and sufficient conditions for M-method to be absolutely regular are:

(i) $\sum_{k=0}^{\infty} a_{n,k}$ converges for all n;

(ii) $\sum_{n=0}^{\infty} \left| \sum_{k=0}^{p} (a_{n,k} - a_{n-1,k}) \right| \leqslant B$

for every p, where B is an absolute constant.

If the matrix elements $a_{n,k} = 0$ for every $k > n$, then the matrix is called triangular matrix.

In particular, if

$$a_{n,k}=\begin{cases}\dfrac{p_{n-k}}{P_n}, & \text{for} \quad k\leqslant n;\\ 0, & \text{for} \quad k>n;\end{cases}$$

where $P_n=\sum_{r=0}^{n} p_r\neq 0$, then t_n defined by (1) is same as the Nörlund mean generat by the sequence of coefficients $\{p_n\}$ usually denoted as (N, p_n) mean.

Similarly, if A_n^{α}, $\alpha>0$, is defined by

$$A_n^{\alpha}=\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{1\cdot 2\cdot 3\cdots n}$$

and

$$a_{n,k}=\begin{cases}\dfrac{A_{n-k}^{\alpha-1}}{A_n^{\alpha}}, & \alpha>0, \quad \text{for} \quad k\leqslant n;\\ 0, & \text{for} \quad k>n;\end{cases}$$

then t_n is same as the (C,α) mean, the familiar Cesàro mean of order α.

2. Notations

Let $f(t)$ be a periodic function with period 2π and integrable L over $(-\pi,\pi)$. Let

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)=\sum_{n=1}^{\infty}A_n(t)$$

be the Fourier series of $f(t)$. The series

$$\sum_{n=1}^{\infty}(b_n\cos nt-a_n\sin nt)=\sum_{n=1}^{\infty}B_n(t)$$

is called the allied series of the Fourier series. In what follows, we use the followi notations:

$$\psi(t)=\tfrac{1}{2}\{f(x+t)-f(x-t)\};$$

$$\Psi_0(t)=\psi(t);$$

$$\Psi_{\alpha}(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-u)^{\alpha-1}\psi(u)\,du, \quad \alpha>0, \quad t>0;$$

$$\psi_{\alpha}(t)=\Gamma(1+\alpha)t^{-\alpha}\Psi_{\alpha}(t), \quad \alpha\geqslant 0, \quad t>0;$$

$$\bar{g}(n,t)=\frac{2}{\pi}\sum_{k=1}^{n}\frac{k+1}{k}a_{n,k}\cos kt;$$

$$\bar{J}(n,t)=\frac{1}{\Gamma(1-\alpha)}\int_u^{\pi}(t-u)^{-\alpha}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\bar{g}(n,t)\right\}\mathrm{d}t;$$

$$\Delta u_n=u_n-u_{n+1};$$

$$\tau=[1/t];$$

and $[x]$ denotes the largest integer $\leqslant x$; and B is taken for absolute constants, not necessarily the same in each occurrence.

3. Theorem

Bosanquet and Hyslop [1] proved the following theorem on the absolute Cesàro summability of the conjugate series.

Theorem 1. *If* $0<\alpha<1$, $\psi_\alpha(+0)=0$ *and*

$$\int_0^{\pi}\frac{|\mathrm{d}\psi_\alpha(t)|}{t^\alpha}<\infty,$$

then the allied series (conjugate series) is summable $|C,\beta|$ *at* $t=x$, *for* $\beta>\alpha$.

Further Kishore and Hotta [5] generalized the theorem due to Kishore [4] by removing the restriction of $\{p_n\}$ being non-increasing. They proved:

Theorem 2. *If* $0<\alpha<1$, $\Psi_\alpha(+0)=0$ *and*

$$\int_0^{\pi}\frac{|\mathrm{d}\Psi_\alpha(u)|}{u^\alpha}<\infty$$

and $\{p_n\}$ *is non-negative sequence, such that*

(i) $$\{V_n\}=\left\{\frac{(n+1)p_n}{P_n}\right\}\in\text{b.v.}$$

and

(ii) $$\frac{P_k}{k^\alpha}\sum_{n=k}^{\infty}\frac{n^\alpha}{(n+1)P_n}\leqslant B,$$

where B *is an absolute constant independent of* k, *then the conjugate series of the Fourier series of* $f(t)$ *is summable* $|N,p_n|$, *at* $t=x$.

We prove:

Theorem. *Let* $|M|=(a_{n,k})$ *be an infinite triangular matrix with* $\lim_{n\to\infty}a_{n,k}=0$. *Further let us write* $\sum_{r=k}^{n}a_{n,r}=A_{n,k}$ *and assume that* $A_{n,0}=1$ *for every* $n\geqslant 0$. *If* $0<\alpha<1$, $\Psi_\alpha(+0)=0$ *and*

$$\int_0^{\pi}\frac{|\mathrm{d}\Psi_\alpha(t)|}{t^\alpha}<\infty$$

then the conjugate series of the Fourier series of $f(t)$ is summable $|M|$ at $t=x$, provided

$$a_{n-1,k} \geqslant a_{n,k+1} \geqslant a_{n,k} \quad \text{for all } n,k \text{ with } \quad k \leqslant n-1 \tag{4}$$

$$(n+1)(a_{n-1,k} - a_{n,k+1}) \geqslant \Delta_k\{(n-k)a_{n,k}\} \tag{5}$$

$$\Sigma \frac{a_{n,0}}{n+1} < \infty \tag{6}$$

and

$$\sup_k \frac{1}{k^\alpha} \sum_{n=k}^{\infty} \frac{A_{n,n-k}}{n^{1-\alpha}} < \infty. \tag{7}$$

4. Lemmas

We require the following Lemmas for the proof of our theorem.

Lemma 1. If $a_{n-1,k} \geqslant a_{n,k+1} \geqslant a_{n,k}$ for all n,k such that $k \leqslant n-1$ and $\lim_{n\to\infty} a_{n,k} = 0$. Then $a_{n,k} \geqslant 0$ for all n,k.

Proof. For $k \leqslant n-1$ we have

$$a_{n,k} = \sum_{r=n+1}^{\infty} (a_{r-1,k} - a_{r,k}) \geqslant 0$$

as $a_{n-1,k} \geqslant a_{n,k}$ for all n, so that $a_{r-1,k} - a_{r,k} \geqslant 0$ for all r,k. Hence $a_{n,k} \geqslant 0$ for all n,k.

Lemma 2. [6] *If $\{a_{n,k}\}_{k=0}^{n}$ is a non-negative and non-decreasing sequence with respect to k, then for $0 \leqslant a < b \leqslant \infty$, $0 \leqslant t \leqslant \pi$ and for every n*

$$\left| \sum_{k=a}^{b} a_{n,n-k} \exp[i(n-k)t] \right| \leqslant BA_{n,n-\tau},$$

where B is an absolute constant and $\tau = [1/t]$.

Lemma 3. Let conditions (4)–(7) of the theorem hold. If further the sequence of partial sums of Σu_n be bounded, then a necessary and sufficient condition for it to be summable $|M|$, is that

$$\sum_{n=1}^{\infty} \frac{1}{n+1} \left| \sum_{k=0}^{n} (k+1) a_{n,k} u_k \right| < \infty.$$

Proof. Since the matrix $(a_{n,k})$ is triangular and $s_k = u_0 + u_1 + \cdots + u_k$, we have

$$t_n = \sum_{k=0}^{n} a_{n,k} s_k$$

$$= \sum_{k=0}^{n} A_{n,k} u_k.$$

Hence

$$t_n - t_{n-1} = \sum_{k=0}^{n} (A_{n,k} - A_{n-1,k})u_k$$

$$= \sum_{k=0}^{n} (A_{n,k} - A_{n-1,k} - \frac{k+1}{n+1} a_{n,k})u_k + \frac{1}{n+1} \sum_{k=0}^{n} (k+1)a_{n,k}u_k$$

$$= \Sigma_1 + \Sigma_2, \quad \text{say.}$$

Thus, in order to prove the Lemma, it is sufficient to show that under our hypothesis,

$$\sum_{n=1}^{\infty} |\Sigma_1| < \infty.$$

Now,

$$|\Sigma_1| = \left| \sum_{k=0}^{n} (A_{n,k} - A_{n-1,k})u_k - \sum_{k=0}^{n} \frac{k+1}{n+1} a_{n,k} u_k \right|$$

$$= \left| \sum_{k=0}^{n-1} \Delta_k (A_{n,k} - A_{n-1,k}) \sum_{r=0}^{k} u_r - \frac{1}{n+1} \sum_{r=0}^{n-1} \Delta_k \{(k+1)a_{n,k}\} \sum_{r=0}^{k} u_r \right|$$

$$= \left| \sum_{k=0}^{n-1} (a_{n,k} - a_{n-1,k}) s_k - \frac{1}{n+1} \sum_{k=0}^{n-1} \{(k+1)a_{n,k} - (k+2)a_{n,k+1}\} s_k \right|$$

$$= \left| \sum_{k=0}^{n-1} \left[\left(1 - \frac{k+1}{n+1}\right) a_{n,k} - \left(1 - \frac{k+2}{n+1}\right) a_{n,k+1} - (a_{n-1,k} - a_{n,k+1}) \right] s_k \right|$$

$$= \left| \sum_{k=0}^{n-1} \left[\left(\frac{n-k}{n+1} a_{n,k} - \frac{n-k-1}{n+1} a_{n,k+1} \right) - (a_{n-1,k} - a_{n,k+1}) \right] s_k \right|$$

$$\leq \sum_{k=0}^{n-1} \left| \frac{(n-k)a_{n,k} - (n-k-1)a_{n,k+1}}{n+1} - (a_{n-1,k} - a_{n,k+1}) \right| |s_k|$$

$$\leq A \sum_{k=0}^{n-1} \left| \frac{(n-k)a_{n,k} - (n-k-1)a_{n,k+1}}{n+1} - (a_{n-1,k} - a_{n,k+1}) \right|$$

$$\leq A \sum_{k=0}^{n-1} \left[(a_{n-1,k} - a_{n,k+1}) - \frac{(n-k)a_{n,k} - (n-k-1)a_{n,k+1}}{n+1} \right]$$

by using (5)

$$= A \left(\sum_{k=0}^{n-1} a_{n-1,k} - \sum_{k=0}^{n-1} a_{n,k+1} - \frac{n}{n+1} a_{n,0} \right)$$

$$= A \left(A_{n-1,0} - A_{n,0} + a_{n,0} - \frac{n}{n+1} a_{n,0} \right)$$

$$= A \left(a_{n,0} - \frac{n}{n+1} a_{n,0} \right)$$

$$= A \frac{a_{n,0}}{n+1}.$$

Therefore,

$$\sum_{n=1}^{\infty} |\Sigma_1| \leqslant A \sum_{n=1}^{\infty} \frac{a_{n,0}}{n+1}$$
$$< \infty, \quad \text{by (6).}$$

This completes the proof of the Lemma.

Lemma 4. [1] *If* $\alpha > 0$, $\eta > 0$, *then the necessary and sufficient conditions that*

$$\int_0^{\eta} \frac{|d\Psi_{\alpha}(u)|}{u^{\alpha}} < \infty$$

and $\Psi_{\alpha}(+0) = 0$, *are that* $\Psi_{\alpha}(t)$ *and* $\theta_{\lambda}(t)$ *should be of bounded variation in* $(0, \eta)$, *for some positive* λ *where*

$$\theta_{\lambda}(t) = \frac{2}{\pi} \int_0^{\pi} \frac{\psi_{\lambda}(t)}{t} dt.$$

Lemma 5. If

$$\bar{g}(n,t) = \frac{2}{\pi} \sum_{k=1}^{n} \frac{k+1}{k} a_{n,k} \cos kt$$

then

$$\bar{g}(n,t) = \begin{cases} O(1), & \text{for all } t; \\ O(A_{n,n-\tau}), & \text{for } t \geqslant 1/n \quad \text{and} \quad \tau = [1/t] \end{cases}$$

and

$$\frac{d}{dt}\bar{g}(n,t) = \begin{cases} O(n), & \text{for all } t; \\ O(nA_{n,n-\tau}), & \text{for } t \geqslant 1/n. \end{cases}$$

Proof. For all t

$$|\bar{g}(n,t)| = \frac{2}{\pi} \left| \sum_{k=1}^{n} \frac{k+1}{k} a_{n,k} \cos kt \right|$$
$$= \frac{2}{\pi} \left| \sum_{k=1}^{n} \left(1 + \frac{1}{k}\right) a_{n,k} \cos kt \right|$$
$$\leqslant \frac{4}{\pi} \sum_{k=1}^{n} a_{n,k}$$
$$= O(1).$$

For $t \geqslant 1/n$

$$|\bar{g}(n,t)| = \frac{2}{\pi} \left| \sum_{k=1}^{n} \frac{k+1}{k} a_{n,k} \cos kt \right|$$
$$\leqslant \frac{4}{\pi} \max \left| \sum_{} a_{} \cos rt \right|$$

Also

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\bar{g}(n,t)\right| = \frac{2}{\pi}\left|\sum_{k=1}^{n}(k+1)a_{n,k}\sin kt\right|.$$

Hence for all t

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\bar{g}(n,t)\right| \leqslant O(n+1)\sum_{k=1}^{n} a_{n,k}$$
$$= O(n)$$

and for $t \geqslant 1/n$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\bar{g}(n,t)\right| \leqslant \frac{2n}{\pi}\max_{1\leqslant k\leqslant n}\left|\sum_{r=1}^{k} a_{n,r}\sin rt\right|$$
$$= O(nA_{n,n-\tau}).$$

Lemma 6. If

$$\bar{J}(n,u) = \frac{1}{\Gamma(1-\alpha)}\int_{u}^{\pi}(t-u)^{-\alpha}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\bar{g}(n,t)\right\}\mathrm{d}t$$

then

$$|\bar{J}(n,u)| = \begin{cases} O(n^{\alpha}), & \text{for all } u; \\ O(n^{\alpha}A_{n,n-U}), & \text{for } u \geqslant 1/n \quad \text{and} \quad U = [1/u]. \end{cases}$$

Proof.

$$\bar{J}(n,u) = \frac{1}{\Gamma(1-\alpha)}\int_{u}^{\pi}(t-u)^{-\alpha}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\bar{g}(n,t)\right\}\mathrm{d}t$$
$$= \int_{u}^{u+(1/n)} + \int_{u+(1/n)}^{\pi}$$
$$= \bar{J}_1 + \bar{J}_2, \quad \text{say.}$$

For all t

$$|\bar{J}_1| \leqslant \frac{1}{\Gamma(1-\alpha)}\int_{u}^{u+(1/n)}(t-u)^{-\alpha}\left|\frac{\mathrm{d}}{\mathrm{d}t}\bar{g}(n,t)\right|\mathrm{d}t$$
$$= \frac{1}{\Gamma(1-\alpha)}\int_{u}^{u+(1/n)}(t-u)^{-\alpha}O(n)\,\mathrm{d}t$$
$$= O(n)\int_{u}^{u+(1/n)}(t-u)^{-\alpha}\mathrm{d}t$$
$$= O(n^{\alpha}) \quad \text{and}$$

$$|\bar{J}_2| = \frac{1}{\Gamma(1-\alpha)}\left|\int_{u+(1/n)}^{\pi}(t-u)^{-\alpha}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\bar{g}(n,t)\right\}\mathrm{d}t\right|$$
$$= O(n^{\alpha})\left|\int_{u+(1/n)}^{\eta}\frac{\mathrm{d}}{\mathrm{d}t}\bar{g}(n,t)\,\mathrm{d}t\right|, \quad \text{where} \quad u+(1/n) < \eta < \pi$$

$$= O(n^{\alpha})\{|\bar{g}(n,t)|\}_{u+(1/n)}^{\eta}$$
$$= O(n^{\alpha}).$$

For $t \geqslant 1/n$

$$|\bar{J}_1| \leqslant \frac{1}{\Gamma(1-\alpha)} \int_u^{u+(1/n)} (t-u)^{-\alpha} \left| \frac{\mathrm{d}}{\mathrm{d}t} \bar{g}(n,t) \right| \mathrm{d}t$$
$$= \frac{1}{\Gamma(1-\alpha)} \int_u^{u+(1/n)} (t-u)^{-\alpha} O(nA_{n,n-\tau})\, \mathrm{d}t$$
$$= O(nA_{n,n-U}) \int_u^{u+(1/n)} (t-u)^{-\alpha}\, \mathrm{d}t$$
$$= O(n^{\alpha} A_{n,n-U}),$$

and

$$|\bar{J}_2| = \left| \frac{1}{\Gamma(1-\alpha)} \int_{u+(1/n)}^{\pi} (t-u)^{-\alpha} \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \bar{g}(n,t) \right\} \mathrm{d}t \right|$$
$$= \frac{n^{\alpha}}{\Gamma(1-\alpha)} \left| \int_{u+(1/n)}^{\eta} \frac{\mathrm{d}}{\mathrm{d}t} \bar{g}(n,t)\, \mathrm{d}t \right|, \quad \text{where} \quad u + (1/n) < \eta < \pi$$
$$= O(n^{\alpha})\{|\bar{g}(n,t)|\}_{u+(1/n)}^{\eta}$$
$$= O(n^{\alpha} A_{n,n-U}).$$

5. Proof of the Theorem

Let

$$\bar{T}_n = \sum_{k=1}^{n} a_{n,k}(k+1)B_k(x).$$

It is known [2] that if $\psi_{\alpha}(t)$, $0 \leqslant \alpha \leqslant 1$, is of bounded variation, the series conjugate to the Fourier series of $f(t)$ is convergent. Hence, in view of Lemma 3 and Lemma 4 for establishing this theorem we need only to prove that

$$\sum_{n=1}^{\infty} \frac{|\bar{T}_n|}{n+1} < \infty.$$

Now since

$$(k+1)B_k = -\frac{2}{\pi} \int_0^{\pi} \frac{k+1}{k} \psi(t) \frac{\mathrm{d}}{\mathrm{d}t} \cos kt\, \mathrm{d}t.$$

We have

$$\bar{T}_n = -\frac{2}{\pi} \sum_{k=1}^{n} a_{n,k} \int_0^{\pi} \frac{k+1}{k} \psi(t) \frac{\mathrm{d}}{\mathrm{d}t} \cos kt\, \mathrm{d}t$$
$$= -\int_0^{\pi} \psi(t) \frac{\mathrm{d}}{\mathrm{d}t} \bar{g}(n,t)\, \mathrm{d}t;$$

where $\bar{g}$ is defined as in § 2. Again, since $\Psi(+0)=0$, we have

$$\begin{aligned}\bar{T}_n &= -\int_0^\pi \frac{\mathrm{d}}{\mathrm{d}t}\bar{g}(n,t)\left\{\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-u)^{-\alpha}\,\mathrm{d}\Psi_\alpha(u)\right\}\mathrm{d}t \\ &= -\frac{1}{\Gamma(1-\alpha)}\int_0^\pi \mathrm{d}\Psi_\alpha(u)\int_u^\pi (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\bar{g}(n,t)\,\mathrm{d}t \\ &= -\int_0^\pi \bar{J}(n,u)\,\mathrm{d}\Psi_\alpha(u);\end{aligned}$$

where $\bar{J}$ is defined in § 2. Hence

$$\begin{aligned}\sum_{n=1}^\infty \frac{|\bar{T}_n|}{n+1} &= \sum_{n=1}^\infty \frac{1}{n+1}\left|\int_0^\pi \bar{J}(n,u)\,\mathrm{d}\Psi_\alpha(u)\right| \\ &\leqslant \int_0^\pi \frac{|\mathrm{d}\Psi_\alpha(u)|}{u^\alpha}\sum_{n=1}^\infty \frac{1}{n+1}|u^\alpha \bar{J}(n,u)|,\end{aligned}$$

But, since

$$\int_0^\pi \frac{|\mathrm{d}\Psi_\alpha(u)|}{u^\alpha}$$

is finite, we need only to show

$$\sum_{n=1}^\infty \frac{1}{n+1}|u^\alpha \bar{J}(n,u)| < \infty$$

uniformly for all $u, 0<u<\pi$. Now,

$$\begin{aligned}&\sum_{n=1}^\infty \frac{1}{n+1}|u^\alpha \bar{J}(n,u)| \\ &= \sum_{n=1}^{U-1}\frac{1}{n+1}|u^\alpha \bar{J}(n,u)| + \sum_{n=U}^\infty \frac{1}{n+1}|u^\alpha \bar{J}(n,u)| \\ &= \sum_{n<U}\frac{1}{n+1}O(n^\alpha u^\alpha) + \sum_{n\geqslant U}\frac{1}{n+1}O(n^\alpha u^\alpha A_{n,n-U}) \\ &= O(u^\alpha)\sum_{n<U} n^{\alpha-1} + O(u^\alpha)\sum_{n\geqslant U}\frac{A_{n,n-U}}{n^{1-\alpha}} \\ &= O(1)+O(1) \\ &= O(1), \text{ by Lemma 6 and (7) where } U=[1/u].\end{aligned}$$

This completes the proof of the theorem.

6. Remark

If $A_n^\beta, \beta>0$, is as defined in § 1, then we have

$$\frac{A_{n-k-1}^{\beta-1}}{A_{n-1}^\beta} \geqslant \frac{A_{n-k-1}^{\beta-1}}{A_n^\beta} \geqslant \frac{A_{n-k}^{\beta-1}}{A_n^\beta}$$

and

$$\frac{1}{k^{\alpha}} \sum_{n=k}^{\infty} \frac{A_k^{\beta}}{n^{1-\alpha} A_n^{\beta}} < \infty.$$

Thus, Theorem 1 follows as a particular case of our theorem. By setting

$$a_{n,k} = \begin{cases} \dfrac{p_{n-k}}{P_n}, & \text{for} \quad k \leqslant n; \\ 0, & \text{for} \quad k > n; \end{cases}$$

theorem 2 follows from our theorem. Further a preliminary version of our theorem was proved in [3].

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On the $|\bar{N}, p_n|$ summability factors of infinite series

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Abstract. In this paper a theorem of Chen [1] on $|\bar{N}, p_n|$ summability factors of infinite series has been proved under weaker conditions.

Keywords. Absolute summability; summability factors; infinite series.

1. Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n). Let (p_n) be a sequence of positive real constants such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0,\ i \geqslant 1). \tag{1.1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.2}$$

defines the sequence (t_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|$, if (see [2])

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty. \tag{1.3}$$

The series $\sum a_n$ is said to be bounded $[\bar{N}, p_n]$, if (see [2])

$$\sum_{v=1}^{n} p_v |s_v| = O(P_n) \quad \text{as} \quad n \to \infty. \tag{1.4}$$

A sequence (λ_n) is said to convex if (see [3]) $\Delta^2 \lambda_n \geqslant 0$, for all n, where $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$, $\Delta^2 \lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$.

2. Chen's theorem

Chen [1] proved the following theorem.

Theorem A. *Let (p_n) be a positive non-increasing sequence such that $P_n \to \infty$. If $\sum a_n$*

is bounded $[\bar{N}, p_n]$ *and if* (λ_n) *is a convex sequence such that* $\sum p_n\lambda_n$ *is convergent, also*

$$1/n = O(p_n) \tag{1}$$

and

$$\Delta(1/p_n) = O(1) \quad as \quad n \to \infty, \tag{2}$$

then the series $\sum a_n\lambda_n$ *is summable* $|\bar{N}, p_n|$.

3. Main result

The object of the present paper is to obtain a more general theorem than the Theorem A under weaker conditions. Our theorem is as follows.

Theorem. *Let* $\sum a_n$ *be a bounded* $[\bar{N}, p_n]$. *Let* (p_n) *be a positive sequence such that* $P_n \to \infty$ *and there be sequences* (β_n) *and* (λ_n) *such that*

$$|\Delta\lambda_n| \leqslant \beta_n, \tag{3}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{4}$$

$$\sum_{n=1}^{\infty} nP_n|\Delta\beta_n| < \infty, \tag{5}$$

$$P_n|\lambda_n| = O(1) \quad as \quad n \to \infty. \tag{6}$$

If (p_n) *satisfies the condition* (1) *of Theorem A, then the series* $\sum a_n\lambda_n$ *is summable* $|\bar{N}, p_n|$.

Remark. Here we shall justify why we have introduced the sequence (β_n) in statement of the theorem while the condition $\sum_1^\infty nP_n|\Delta^2\lambda_n| < \infty$ serves the purpose in proving the theorem. Also we note that when $(\Delta\lambda_n)$ oscillates it may be possible to choose (β_n) so that $|\Delta\beta_n|$ is significantly smaller than $|\Delta^2\lambda_n|$ so that $\sum_1^\infty nP_n|\Delta\beta_n| < \infty$ is a weaker requirement than $\sum_1^\infty nP_n|\Delta^2\lambda_n| < \infty$. The following example justifies the above argument. Take

$$\Delta\lambda_n = \begin{cases} \dfrac{1}{(n+1)(n+2)}, & (n, \text{even}) \\ 0, & (n, \text{odd}), \end{cases}$$

then

$$\Delta^2\lambda_n = \begin{cases} \dfrac{1}{(n+1)(n+2)}, & (n, \text{even}) \\ \dfrac{-1}{(n+2)(n+3)}, & (n, \text{odd}). \end{cases}$$

But we can take

$$\beta_n = 1/[(n+1)(n+2)],$$

so that

$$\Delta\beta_n = 2/[(n+1)(n+2)(n+3)].$$

Thus

$$\sum_{n=1}^{\infty} nP_n|\Delta\beta_n| < \infty$$

is weaker than

$$\sum_{n=1}^{\infty} nP_n|\Delta^2\lambda_n| < \infty.$$

4. Lemma

We need the following lemma for the proof of our theorem.

Lemma. Under the conditions on $(P_n), (\beta_n)$ and (λ_n) as taken in the statement of the theorem, the following conditions hold, when (5) is satisfied.

$$nP_n\beta_n = O(1) \quad as \quad n \to \infty, \tag{7}$$

and

$$\sum_{n=1}^{\infty} P_n\beta_n < \infty. \tag{8}$$

Proof. Since $\beta_n \to 0$ as $n \to \infty$, by (4), we have

$$\beta_n = \sum_{v=n}^{\infty} \Delta\beta_v.$$

As (nP_n) is increasing,

$$nP_n\beta_n \leqslant \sum_{v=n}^{\infty} vP_v|\Delta\beta_v| = O(1), \quad \text{by (5).}$$

Now,

$$\sum_{n=1}^{\infty} P_n\beta_n \leqslant \sum_{n=1}^{\infty} P_n \sum_{v=n}^{\infty} |\Delta\beta_v| = \sum_{v=1}^{\infty} |\Delta\beta_v| \sum_{n=1}^{v} P_n$$

$$\leqslant \sum_{v=1}^{\infty} vP_v|\Delta\beta_v| < \infty, \quad \text{by (5).}$$

Hence

$$\sum_{n=1}^{\infty} P_n\beta_n < \infty.$$

This completes the proof of the lemma.

5. Proof of the theorem

Let (T_n) be the sequence of the $(\bar{N}, p_n)$ mean of the series $\Sigma a_n\lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r\lambda_r = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1})a_v\lambda_v.$$

Then, for $n \geqslant 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v.$$

Using Abel's transformation, we get

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta(P_{v-1}\lambda_v) s_v + \frac{1}{P_n} p_n s_n \lambda_n \\ &= \frac{1}{P_n} p_n s_n \lambda_n - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \Delta\lambda_v \\ &= T_{n,1} + T_{n,2} + T_{n,3}, \text{ say.} \end{aligned}$$

To prove the theorem we have to show that

$$\sum_{n=1}^{\infty} |T_{n,r}| < \infty, \quad \text{for} \quad r = 1, 2, 3. \tag{9}$$

Firstly, since $1/P_n = O(1)$ as $n \to \infty$, we have

$$\begin{aligned} \sum_{n=1}^{m} |T_{n,1}| &= \sum_{n=1}^{m} \frac{p_n}{P_n} |s_n| \, |\lambda_n| = O(1) \sum_{n=1}^{m} |\lambda_n| p_n |s_n| = O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} p_v |s_v| \\ &\quad + O(1)|\lambda_m| \sum_{n=1}^{m} p_n |s_n| = O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| P_n + O(1)|\lambda_m| P_m \\ &= O(1) \sum_{n=1}^{m-1} P_n \beta_n + O(1)|\lambda_m| P_m = O(1) \quad \text{as} \quad m \to \infty, \end{aligned}$$

by (3), (6), (8) and the hypothesis of the theorem. As in $T_{n,1}$, we get

$$\begin{aligned} \sum_{n=2}^{m+1} |T_{n,2}| &= \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left| \sum_{v=1}^{n-1} p_v s_v \lambda_v \right| \leqslant \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v |s_v| \, |\lambda_v| \\ &= \sum_{v=1}^{m} p_v |s_v| \, |\lambda_v| \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = O(1) \sum_{v=1}^{m} p_v |s_v| \, |\lambda_v| \\ &= O(1) \quad \text{as} \quad m \to \infty. \end{aligned}$$

Finally, we get

$$\begin{aligned} \sum_{n=2}^{m+1} |T_{n,3}| &= \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left| \sum_{v=1}^{n-1} s_v P_v \Delta\lambda_v \right| \leqslant \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v |\Delta\lambda_v| \, |s_v| \\ &= \sum_{v=1}^{m} P_v |\Delta\lambda_v| \, |s_v| \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = O(1) \sum_{v=1}^{m} |\Delta\lambda_v| \, |s_v| \\ &= O(1) \sum_{v=1}^{m} \beta_v |s_v|, \quad \text{by (3).} \end{aligned}$$

nce, using the fact that $1/v = O(p_v)$, by (1), we have that

$$\sum_{n=2}^{m+1} |T_{n,3}| = O(1) \sum_{v=1}^{m} \frac{1}{v} v\beta_v |s_v| = O(1) \sum_{v=1}^{m} v\beta_v p_v |s_v|$$

$$= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} p_r |s_r| + O(1) m\beta_m \sum_{v=1}^{m} p_v |s_v|$$

$$= O(1) \sum_{v=1}^{m-1} v|\Delta\beta_v| P_v + O(1) \sum_{v=1}^{m-1} P_v \beta_{v+1} + O(1) m\beta_m P_m$$

$$= O(1) \quad \text{as} \quad m \to \infty,$$

y (5), (7), (8) and hypothesis of the theorem. Therefore, we get

$$\sum_{n=1}^{m} |T_{n,r}| = O(1) \quad \text{as} \quad m \to \infty, \quad \text{for} \quad r = 1, 2, 3.$$

This completes the proof of the theorem.

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On the bilateral generating functions of simple Bessel polynomial

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Abstract. In this note the bilateral generating relation of simple Bessel polynomial derived by Das, Chongdar and Chatterjea has been extended by using the group-theoretic method. An application of our result is also pointed out.

Keywords. Bilateral generating function; Bessel polynomial; group-theoretic method.

1. Introduction

The simple Bessel polynomial is defined by

$$y_n(x) = {}_2F_0(-n, n+1; -; -x/2), \quad n > 0.$$

In this note we have derived the following theorem on generating functions of the said special function as the main result of our discussion.

Theorem 1. *If*

$$G(x, w) = \sum_{n=0}^{\infty} a_n y_{n+m}(x) \frac{w^n}{n!}, \tag{1}$$

where m is a non-negative integer, then

$$(1-2wx)^{-(m+1)/2} \exp\left(\frac{1-(1-2wx)^{\frac{1}{2}}}{x}\right) G\left(\frac{x}{(1-2wx)^{\frac{1}{2}}}, \frac{wy}{(1-2wx)^{\frac{1}{2}}}\right) \tag{2}$$

$$= \sum_{n=0}^{\infty} \frac{w^n}{n!} f_n(y) y_{n+m}(x),$$

where

$$f_n(y) = \sum_{p=0}^{n} a_p \binom{n}{p} y^p.$$

The importance of the above theorem lies in the fact that whenever one knows a generating relation of the form (1) the corresponding bilateral generating relation can at once be written down from (2). Thus one can get a large number of bilateral generating relations by attributing different values to a_n.

2. Proof of the theorem

We first define the linear partial differential operator R as

$$R = x^2 z \frac{\partial}{\partial x} + xz^2 \frac{\partial}{\partial z} + z\{1 + (m+1)x\}$$

such that

$$R(F_{n+m}(x,z)) = F_{n+m+1}(x,z), \tag{3}$$

where

$$F_{n+m}(x,z) = y_{n+m}(x)z^n.$$

The extended form of the group generated by R is

$$\exp(wR)f(x,z) = (1-2wxz)^{-(m+1)/2} \exp\left(\frac{1-(1-2wxz)^{\frac{1}{2}}}{x}\right) \times f\left(\frac{x}{(1-2wxz)^{\frac{1}{2}}}, \frac{z}{(1-2wxz)^{\frac{1}{2}}}\right). \tag{4}$$

Let us consider the following generating relation:

$$G(x,w) = \sum_{n=0}^{\infty} a_n y_{n+m}(x) \frac{w^n}{n!}.$$

Replacing w by wyz we get

$$G(x,wyz) = \sum_{n=0}^{\infty} a_n \frac{(wy)^n}{n!} F_{n+m}(x,z). \tag{5}$$

Now operating both sides by $\exp(wR)$ we get

$$(\exp wR)G(x,wyz) = (\exp wR) \sum_{n=0}^{\infty} a_n \frac{(wy)^n}{n!} F_{n+m}(x,z).$$

The left hand member of the above expression is

$$(1-2wxz)^{-(m+1)/2} \exp\left(\frac{1-(1-2wxz)^{\frac{1}{2}}}{x}\right) G\left(\frac{x}{(1-2wxz)^{\frac{1}{2}}}, \frac{wyz}{(1-2wxz)^{\frac{1}{2}}}\right), \tag{6}$$

and the right hand member of the said expression is

$$\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} a_n \frac{(wy)^n}{n!} \frac{w^p}{p!} R^p(F_{n+m}(x,z))$$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} a_n \frac{y^n}{n!} \frac{w^{n+p}}{p!} F_{n+m+p}(x,z)$$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{n} a_{n-p} \frac{y^{n-p}}{(n-p)!} \frac{w^n}{p!} F_{n+m}(x,z)$$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{n} a_{n-p} \frac{(wz)^n}{p!} \frac{y^{n-p}}{(n-p)!} y_{n+m}(x)$$

$$= \sum_{n=0}^{\infty} \frac{(wz)^n}{n!} f_n(y) y_{n+m}(x), \tag{7}$$

where

$$f_n(y) = \sum_{p=0}^{n} a_p \binom{n}{p} y^p.$$

Equating (6) and (7) and then putting $z = 1$ we get

$$(1 - wx)^{-(m+1)/2} \exp\left(\frac{1-(1-2wx)^{\frac{1}{2}}}{x}\right) G\left(\frac{x}{(1-2wx)^{\frac{1}{2}}}, \frac{wy}{(1-2wx)^{\frac{1}{2}}}\right)$$

$$= \sum_{n=0}^{\infty} \frac{w^n}{n!} f_n(y) y_{n+m}(x),$$

where

$$f_n(y) = \sum_{p=0}^{n} a_p \binom{n}{p} y^p.$$

This completes the proof of the theorem. It may be mentioned that the above theorem is also found derived in [1] by the classical method.

Now putting $m = 0$ we get

COROLLARY 1

If

$$G(x, w) = \sum_{n=0}^{\infty} a_n y_n(x) \frac{w^n}{n!}$$

then

$$(1-2wx)^{-1/2} \exp\left(\frac{1-(1-2wx)^{\frac{1}{2}}}{x}\right) G\left(\frac{x}{(1-2wx)^{\frac{1}{2}}}, \frac{wy}{(1-2wx)^{\frac{1}{2}}}\right)$$

$$= \sum_{n=0}^{\infty} \frac{w^n}{n!} f_n(y) y_n(x),$$

where

$$f_n(y) = \sum_{p=0}^{n} a_p \binom{n}{p} y^p,$$

which is derived in [2, 3].

3. Application

As an application we consider the following generating relation [4]

$$(1 - 2xw)^{-(m+1)/2} \exp\left(\frac{1-(1-2xw)^{\frac{1}{2}}}{x}\right) y_n\left(\frac{x}{(1-2xw)^{\frac{1}{2}}}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{\cdot} y_{n+m}(x) w^n. \tag{8}$$

Taking $a_n = 1$ we get

$$G(x, w) = (1 - 2xw)^{-(m+1)/2} \exp\left(\frac{1-(1-2xw)^{\frac{1}{2}}}{x}\right) y_n\left(\frac{x}{(1-2xw)^{\frac{1}{2}}}\right).$$

By applying our theorem we get

$$(1-2xw-2xyw)^{-(m+1)/2} \exp\left(2-(1-2xw)^{\frac{1}{2}}-(1-2xw-2xyw)^{\frac{1}{2}}\right)$$
$$\times y_n(x/(1-2xw-2xyw)^{\frac{1}{2}})$$
$$= \sum_{n=0}^{\infty} \frac{w^n}{n!} f_n(y) y_{n+m}(x), \tag{9}$$

where

$$f_n(y) = \sum_{p=0}^{n} \binom{n}{p} y^p.$$

Note. It is of interest to mention that the result (9) at $m = 0$ is obtained by applying Corollary 1 on (8) at $m = 0$.

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Nonlinear random elliptic boundary value problem

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Abstract. In this paper we obtain a solvability result for random elliptic boundary value problem. Our result extends Browder's [2] main theorem to the random case. We use author's [4] fixed point theorem for this purpose.

Keywords. Elliptic boundary value; fixed point theorem.

1. Introduction

In this paper we extend Browder's [2] main result to obtain solvability of random variational boundary value problem involving system of the form

$$A(w)x = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(w, t, x, \ldots, D^m x), \qquad (*)$$

where the coefficients A_α are random having at most polynomial growth.

By using author's [4] random fixed point theorem, we first obtain an abstract result for a class of random operator equation of the type

$$G(w; x(w), x(w)) = z.$$

Here $G: \Omega \times Y \times X \to X^*$ is a random operator, Ω a probability measure space, X, Y the appropriate Banach spaces and $z \in X^*$ a given element. This result is then applied to discuss the solvability of boundary value problem involving system of the form (*).

The abstract result is of interest by itself. A similar result has been obtained by Itoh [3] but with relatively strong conditions on the operator G. We therefore discuss it separately in § 3. In § 4 we consider the random elliptic boundary value problem which is followed by a concrete example. Section 2 gives preliminary definitions concerning monotone operators and random solutions.

2. Preliminaries

Throughout this paper (Ω, β, μ) denotes a probability measure space. For $x \in X$ and $y \in X^*$, $\langle y, x \rangle$ will denote the duality pairing between X and X^*. If X is Hilbert space $\langle\ ,\ \rangle$ indicates the inner product in X. In all our considerations Banach spaces are assumed to be real and separable.

A mapping $g: \Omega \to X$ is said to be measurable, if for any closed subset C of X, $g^{-1}(C)$ is in β. Any X-valued measurable function is called a random variable. If Y is a

Banach space and D a subset of X, $T:\Omega \times D \to Y$ is called a random operator if for any $x \in D$, $T(\cdot)\, x$ is measurable. An equation of the type $T(\cdot)x(\cdot) = y(\cdot)$, where y, a random variable, is called a random operator equation. An X-valued random variable $x(w)$ which satisfies $\mu\{w: T(w)x(w) = y(w)\} = 1$, is said to be random solution of the above equation.

An operator $T:X \to Y$ is said to be bounded if it maps bounded sets into bounded sets. T is demicontinuous if $x_n \to x$ implies that Tx_n converges weakly to T_x. T is said to be compact if it maps bounded sets into precompact sets. $T:X \to X^*$ is called monotone if $\langle Tx - Ty, x - y \rangle \geqslant 0$ for all $x, y \in X$. With these definitions we have the following theorems.

Theorem 2.1 [2]. *Let G be a demicontinuous mapping of X into X^* and let there exists a continuous function $c(r)$ with $c(r) > 0$ for $r > 0$ and $\lim_{r \to \infty} c(r) = \infty$ such that*

$$\langle Gu - Gv, u - v \rangle \geqslant c(\|u - v\|)\|u - v\| \quad \text{for all } u, v \text{ in } X.$$

Then

(a) *G maps X onto X^*.*
(b) *G is 1-1, G^{-1} is defined on all of X^* and is bounded.*
(c) *G^{-1} is continuous from X^* to X and there exists a real valued monotone non-decreasing function $h(r)$ such that*

$$\|G^{-1}y - G^{-1}y_1\| \leqslant h(\|y - y_1\|) \quad \text{for } y, y_1 \text{ in } X^*.$$

In the following B_r is the closed ball $\{x: \|x\| \leqslant r\}$, ∂B_r its boundary with interior $\mathring{B}_r$.

Theorem 2.2 [4]. *Let (Ω, β, μ) be a probability measure space and X a separable reflexive Banach space, $T:\Omega \times B_r \to X$ a compact and continuous random operator. Suppose for a fixed $w \in \Omega$, $T(w)$ satisfies the condition:*
($\leqslant$) If for some $x \in \partial B_r$ the operator equation $T(w)x = \alpha x$ holds, then $\alpha < 1$. Then there exists a random solution $x(w)$ in $\mathring{B}_r$ of the operator equation $T(w)x(w) = x(w)$.

3. Abstract theorem

Theorem 3.1. *Let X be a separable reflexive Banach space with dual X^*. Let Y be a second Banach space with $X \subset Y$ and $\|x\|_Y \leqslant \|x\|_X$ for all $x \in X$. Let $G:\Omega \times Y \times X \to X^*$ be a random mapping satisfying the following conditions:*

(a) *For every $w \in \Omega$, there exists a continuous mapping $c(w; r)$ with $\lim_{r \to \infty} c(w, r) = \infty$ and $c(w; r) > 0$ for $r > 0$ and a constant $R > 0$ such that*

$$\langle G(w; y, x_1) - G(w; y, x_2), x_1 - x_2 \rangle \geqslant c(w; \|x_1 - x_2\|)\|x_1 - x_2\|_X$$

almost everywhere for all $x_1, x_2 \in X$ and $y \in Y$ with $\|y\|_Y \leqslant R$.
(b) *$\langle G(w; x, kx) - z, x \rangle > 0$ for all $x \in X$ with $\|x\|_X = R$ and every $k \geqslant 1$ ($z \in X^*$ a fixed element).*
(c) *For each $w \in \Omega$, $x \to G(w; y, x)$ is demicontinuous for fixed $y \in Y$ and $y \to G(w; y, x)$ is compact and continuous for fixed $x \in X$.*

Then there exists a random solution $x(w) \in X$ of the equation $G(w, x(w)x(w)) = z$.

Proof. Let $y \in X$ be fixed such that $\|y\|_X \leqslant R$. Then $y \in Y$ with $\|y\|_Y \leqslant R$ and in view of condition (a) of the theorem it follows that the random mapping $G(w; y, \cdot)$ satisfies the hypothesis of Theorem 1.1 for a fixed $w \in \Omega$. Hence it implies that $G(w; y, \cdot)$ is 1-1 with $G^{-1}(w; y, \cdot)$ defined and continuous on all of X^*. By Nashed–Salehi theorem [5] it now follows that $G^{-1}(w; y, \cdot)$ is also random, i.e. given $z \in X^*$ there exists a unique random solution $x(w)$ of the equation $G(w; y, x(w)) = z$. Denoting $T(w)y$ we get a mapping $T(w): B_R \to X$ such that $G(w; y, T(w)y) = z$. $T(w)$ is random as $T(w)y$ is a random variable for every $y \in B_R$.

We now claim that for every $w \in \Omega$, $T(w)$ is a compact and continuous mapping satisfying the condition ($\leqslant$) of Theorem 1.2. We first observe that $T(w)$ is bounded. This can be seen from the following inequality and the fact that $Y \to G(w; y, x)$ is compact and hence bounded for every $x \in X$.

$$\begin{aligned} c(w; \|T(w)y\|)\|T(w)y\| &\leqslant \langle G(w; y, T(w)y) - G(w; y, 0), T(w)y - 0 \rangle \\ &= \langle z - G(w; y, 0), T(w)y \rangle \\ &\leqslant \|z - G(w; y, 0)\| \, \|T(w)y\|, \end{aligned}$$

which gives

$$c(w; \|T(w)y\|) \leqslant \|z - G(w; y, 0)\|.$$

We now proceed to show that $G(w)$ is compact. Let $\|y_n\| \leqslant R$ be any sequence. Since $T(w)$ is bounded and X is reflexive, $T(w)y_n$ has a subsequence $T(w)y_{n_k}$ converging weakly to $x \in X$. We will show that this convergence is strong.

Since $y \to G(w; y, x)$ is compact for every $x \in X$, it follows that $G(w; y_{n_k}, x)$ has a subsequence, which we will not differentiate, converging to $z_0 \in X^*$. Consider

$$\begin{aligned} &c(w; \|T(w)y_{n_k} - x\|)\|T(w)y_{n_k} - x\| \\ &\quad \leqslant \langle G(w; y_{n_k}, T(w)y_{n_k}) - G(w; y_{n_k}, x), T(w)y_{n_k} - x \rangle \\ &\quad = \langle z - z_0 + (z_0 - G(w; y_{n_k}, x)), T(w)y_{n_k} - x \rangle \\ &\quad \leqslant \langle z - z_0, T(w)y_{n_k} - x \rangle + \|z_0 - G(w; y_{n_k}, x)\| \, \|T(w)y_{n_k} - x\|. \end{aligned}$$

Since $T(w)y_{n_k} - x \to 0$ and $z_0 - G(w; y_{n_k}, x) \to 0$, the right hand side of the above inequality goes to zero as $n \to \infty$ and hence $\|T(w)y_{n_k} - x\| \to 0$.

One can similarly prove that the mapping $T(w)$ is continuous. To prove that $T(w)$ satisfies condition ($\leqslant$), let $T(w)y = \alpha y$ for some y in ∂B_R. Then $G(w; y, T(w)y) = G(w; y, \alpha y)$. But $G(w; y, T(w)y) = z$, so we have $\langle G(w; y, \alpha y), y \rangle - \langle z, y \rangle = 0$. That is, $\langle G(w; y, \alpha y) - z, y \rangle = 0$ for $y \in \partial B_R$. In view of the assumption (b) on G, it implies that $\alpha < 1$.

Thus the random operator $T(w)$ satisfies all the conditions of Theorem 1.2 and hence there exists a random solution $x(w)$ of the fixed point equation $T(w)x(w) = x(w)$. Using the definition of $T(w)$, it follows that $x(w)$ is a random solution of $G(w; x(w), x(w)) = z$.

Remark. In [3] Itoh has proved a similar result (Theorem 3.4). He has monotonicity condition on the map $x \to G(w; y, x)$ for each fixed $y \in Y$. Our condition (a) is similar to this but it needs to be satisfied only for $y \in Y$ with $\|y\| \leqslant R$. Further, he assumed

complete continuity on $y \to G(w; y, x)$ for each fixed $x \in X$. But we impose only compactness and continuity on this map. We feel that this is an improvement.

4. Random elliptic boundary value problems

As an immediate application of the above result we now discuss the solvability of elliptic boundary value problem for system of the form

$$A(w)x = \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(w, t, x, \ldots, D^m x). \tag{1}$$

Following Browder [1] we make use of the following terminology. The points of the open set $\Sigma \subset R^n$ will be denoted by $t = (t_1, t_2, \ldots, t_n)$ and the elementary differential operator

$$D^{\alpha} = \prod_{j=1}^{n} \left(\frac{\partial}{\partial t_j} \right)^{\alpha_j}$$

for an ordered n-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of non-negative integers with the order of the operator D^{α} being written as

$$|\alpha| = \sum_{j=1}^{n} \alpha_j \cdot R^{sm}$$

will denote the vector space whose elements are $\xi = \xi_{\alpha} : |\alpha| \leqslant m$ and for any $\xi \in R^{sm}$ and for

$$1 \leqslant p < \infty, \quad |\xi|^p = \sum_{|\alpha| \leqslant m} |\xi_{\alpha}|^p.$$

We define

$$w^{m,p}(\Sigma) = \{x : x \in L^p(\Sigma), D^{\alpha} x \in L^p(\Sigma), |\alpha| \leqslant m\}.$$

$W^{m,p}$ is a reflexive separable Banach space with respect to the norm

$$\| x \|_{m,p} = \left(\sum_{|\alpha| \leqslant m} \int_{\Sigma} |D^{\alpha} x|^p \, dt \right)^{1/p}$$

(x, y) will denote the natural pairing between $x \in L^p(\Sigma)$ and $y \in L^p(\Sigma)$ with $q = p(p-1)^{-1}$.

We assume the following growth condition on each A_{α}.

Assumption 1. $|A_{\alpha}(w; t, \xi)| \leqslant c_{\alpha}(w) |\xi|^{p-1} + g_{\alpha}(w; t)$ almost surely, $c_{\alpha}(w) > 0$, $g_{\alpha}(w) \in L^q$.

Corresponding to (1) we may define the nonlinear Dirichlet form $\alpha(w; x, y)$ for each pairing $x, y \in W^{m,p}(\Sigma)$ by

$$\alpha(w; x, y) = \sum_{|\alpha| \leqslant m} (A_{\alpha}(w; t, x, \ldots, D^m x), D^{\alpha} y).$$

Because of Assumption 1, it follows that for a fixed $w \in \Omega$, $\alpha(w; x, y)$ is well defined for all $x, y \in W^{m,p}(\Sigma)$ and satisfies an inequality of the type

$$|\alpha(w; x, y)| \leqslant g(w; \| x \|_{m,p}) \| y \|_{m,p},$$

where $g(w,r)$ is the function of the real variable r for a fixed $w\in\Omega$. Since $A_\alpha(w)$ is random it follows that $\alpha(w;x,y)$ is a random Dirichlet form.

In the following we assume that there exists a close subspace V of $W^{m,p}(\Sigma)$ with $C_0^\infty(\Sigma)\subset V$.

DEFINITION 4.1

Let f be an element of V^*. Then $x(w)$ is said to be the random solution of the variational boundary value problem for $A(w)x=f$ satisfying the null boundary conditions corresponding to the space V if

(a) $x(w)\in V$ and $x(w)$ is random and
(b) $\alpha(w;x(w),y)=(f,y)$ for all $y\in V$, almost everywhere.

To separate the dependence of Dirichlet form on the lower order from its dependance on the higher order we define a new Dirichlet form $\alpha(w;x,y,z)$ as

$$\alpha(w;x,y,z)=\sum_{|\alpha|\leqslant m}(A_\alpha(w;t,x,\ldots,D^{m-1}x,D^m y),D^\alpha z).$$

Again it follows from Assumption 1 that $\alpha(w;x,y,z)$ is well defined for all $x\in W^{m-1,p}(\Sigma)$ and all y,z in $W^{m,p}(\Sigma)$. By the Sobolev imbedding theorem, $W^{m,p}$ is contained in $W^{m-1,p}$ and the imbedding is compact and

$$|\alpha(w;x,y,z)|\leqslant g(w;\|x\|_{m-1,p})\|y\|_{m,p}\|z\|_{m,p}.$$

Theorem 4.1. *Let $A(w)$ be a system of random differential operator of the form* (1) *satisfying Assumption 1 for a given value of $p>1$. Let V be a closed subspace of $W^{m,p}(\Sigma)$ such that $C_0^\infty(\Sigma)\subset V$. Let the Dirichlet form $\alpha(w;x,y,z)$ be as defined before for $x\in W^{m-1,p}(\Sigma)$ and y,z in $W^{m,p}(\Sigma)$. Suppose all the following conditions are satisfied:*

(a) *For every $w\in\Omega$, there exists a continuous mapping $c(w;r)$ with $\lim_{r\to\infty}c(w,r)=\infty$ and $c(w;r)>0$ for $r>0$ and a constant $R>0$ such that*

$$\alpha(w;x,y,y-z)-\alpha(w;x,z,y-z)\geqslant c(w;\|y-z\|_{m,p})\|y-z\|_{m,p}$$

almost everywhere for all y,z in $W^{m,p}$ and $\|x\|_{n-1,p}\leqslant R$.
(b) *$\alpha(w;x,kx-f,x)>0$ almost everywhere for all x in V with $\|x\|_{m,p}=R$ and $k\geqslant 1$.*

Then for every $f\in V^*$, there exists a random solution $x(w)$ in V of the variational boundary value problem for $A(w)x=f$.

Proof. Let X be the closed subspace V of $W^{m,p}$ with the Banach space structure endowed by $W^{m,p}$. Let Y be the closure of V in $W^{m-1,p}$ with the norm of $W^{m-1,p}$. Now consider the form $\alpha(w;x,y,z)$ defined as before. Then we have

$$|\alpha(w;x,y,z)|\leqslant g(w;\|x\|_Y)\|y\|_X\|z\|_X.$$

For a fixed x,y, $\alpha(w;x,y,z)$ is a continuous random functional on X. Hence by Riesz representation theorem it follows that there exists a unique element $G(w;x,y)\in X$ such that $\alpha(w;x,y,z)=\langle G(w;x,y),z\rangle$ for all $w\in\Omega$ and $z\in X$.

Since $\alpha(w;x,y,z)$ is random, it follows that G is random. In order to apply Theorem

3.1, we need only verify conditions (a), (b) and (c). For a fixed $w\in\Omega$, the verification of these conditions is similar to that of Browder [2] for the deterministic analogue and hence we omit the details.

So it follows that there exists a random solution $x(w)$ of the equation $G(w; x(w), x(w)) = f$. This in turn implies that $\alpha(w; x(w), x(w), z) = (f, z)$ for all $z\in V$. This proves the theorem.

Example. As a concrete example we consider the following elliptic boundary value problem:

$$\frac{\partial}{\partial t_1}\left(b_1(w;t)\frac{\partial x}{\partial t_1}\right)+\frac{\partial}{\partial t_2}\left(b_2(w;t)\frac{\partial x}{\partial t_2}\right)=A_0(w;t,x(t)) \tag{2}$$

$$x(t)=0, \quad t\in\partial\Sigma; \quad t=(t_1,t_2), \quad \Sigma=(0,1)\times(0,1).$$

In order to apply Theorem 3.1, we take $p=2$, $V=W_0^{1,2}(\Sigma)$ and assume that the functions $b_1(w;t), b_2(w;t), A_0(w;t,\xi)$ are random functions and A_0 satisfies the Caratheodory conditions and a growth condition of the type

$$|A_0(w;t,\xi)|\leqslant b(w)|\xi|+g(w;t), \quad b(w)>0 \quad \text{and} \quad g(w;t)\in L^2(\Sigma).$$

Further, $b_1(w;t), b_2(w;t)\in L^2(\Sigma)$ and are such that $b_1(w;t), b_2(w;t)\geqslant r>0$ almost surely. In terms of our earlier notation, the Dirichlet form $\alpha(w;x,y,z)$ is given by

$$\alpha(w;x,y,z)=(A_0(w;t,x(t)),z(t))+\left(b_1(w;t)\frac{\partial y}{\partial t_1},\frac{\partial z}{\partial t_1}\right)+\left(b_2(w;t)\frac{\partial y}{\partial t_2},\frac{\partial z}{\partial t_2}\right).$$

It is easily seen that

$$\begin{aligned}
&\alpha(w;x,y,y-z)-\alpha(w;x,z,y-z)\\
&=\int_\Sigma\left[b_1(w;t)\left(\frac{\partial y}{\partial t_1}-\frac{\partial z}{\partial t_1}\right)^2+b_2(w;t)\left(\frac{\partial y}{\partial t_2}-\frac{\partial z}{\partial t_2}\right)^2\right]\mathrm{d}t\\
&\geqslant r\int_\Sigma\left[\left(\frac{\partial y}{\partial t_1}-\frac{\partial z}{\partial t_1}\right)^2+\left(\frac{\partial y}{\partial t_2}-\frac{\partial z}{\partial t_2}\right)^2\right]\mathrm{d}t\\
&\geqslant r\left(\frac{2\pi^2}{1+2\pi^2}\right)\|y-z\|_{1,2}^2 \quad \text{for all} \quad x\in L^2(\Sigma)
\end{aligned}$$

(by Poincare inequality).

Also assume that there exists an $R>0$ such that A_0 satisfies the following inequality

$$\int_\Sigma A_0(w;t,x(t))x(t)\,\mathrm{d}t\geqslant-\alpha\int_\Sigma x^2(t)\,\mathrm{d}t \quad \text{for} \quad \|x\|_{1,2}=R,$$

(α is some positive constant).

Then we have

$$\alpha(w;x,kx,x)\geqslant-\alpha\|x\|_{L^2}^2+\left(\frac{2kr\pi^2}{1+2\pi^2}\right)\|x\|_{1,2}^2$$

$$\geqslant \left(-\alpha + \frac{2r\pi^2}{1+2\pi^2} \right) k \| x \|_{1,2}^2$$

$$> 0 \quad \text{for} \quad \| x \|_{1,2} = R, \quad k \geqslant 1,$$

provided

$$\alpha < \frac{2r\pi^2}{1+2\pi^2}.$$

Since all the conditions of Theorem 4.1 are satisfied, it follows that there exists a random solution $x(w)$ of the variational boundary value problem corresponding to equation (2).

Acknowledgement

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Unsteady state heat flow in epidermis and dermis of a human body

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Abstract. We investigate the unsteady state temperature distribution in human skin where subcutaneous tissues are not present. The mathematical model is employed for a one-dimensional unsteady state case, taking the blood mass flow rate and metabolic heat generation variable with respect to the position in the dermis. The metabolic heat generation depends on the tissue temperature. The thermal conductivity is taken constant but different in two layers. The problem has been solved using Laplace transform and Bessel functions. Numerical results for a simple case are discussed.

Keywords. Thermal conductivity; mass blood flow rate; metabolic heat generation; Laplace transform; Bessel function.

1. Introduction

The skin and subdermal tissues (SST) region of the body undergoes temperature variation due to changes in environmental conditions and physiological parameters to maintain a uniform body core temperature. The physiological parameters which play a significant role in this process are the rates of metabolism and blood mass flow, perspiration and the structure of the SST region. The skin consists of two layers, epidermis and dermis (córium). The epidermis is composed of a living stratum malpighi which rests upon the dermis, and a dead, horny, superficial stratum corneum. The outer dead layer of the epidermis stratum corneum is composed of flattened cells. The dermis is composed of matted masses of connective tissues and elastic fibres through which pass numerous blood vessels, lymphatics and nerves. The subdermal tissues contain fat cells, connective tissues, blood vessels, lymphatics and nerves. The population density of blood vessels in the dermis (there are no blood vessels in the epidermis) is very thin near the interface of epidermis and dermis, but increases gradually becoming almost uniform in the subdermal part. This gives us some idea about the variation of physical quantities like the rate of blood mass flow, the rate of metabolic heat generation and the thermal conductivity of tissue in this region in relation to its position.

In this study we confine ourselves to those regions of the human body which have almost no subdermal tissues. In view of the properties of SST we take the rates of blood mass flow and metabolic heat generation negligible in the epidermis and linearly variable with respect to the position in the dermis. In addition, the rate of metabolic heat generation is assumed to be temperature-dependent, and varies linearly as $(T_b - T)$ in the dermis. The other physical quantities like tissue density, specific heat of the tissue and blood, and the tissue thermal conductivity are assumed to be constant. It is also assumed that the outer skin surface is exposed to the atmosphere and the heat loss is due to convection, radiation and evaporation of sweat.

2. Problem formulation

If T denotes the temperature at time t and at a point x, then the governing partial differential equation is given by

$$\text{Div}[K \text{ grad } T] + m_b c_b(T_b - T) + S = \rho \bar{c}(\partial T/\partial t), \tag{1}$$

and

$$K(\partial T/\partial x) = h(T - T_a) + LE \tag{2}$$

at the skin surface where K is the thermal conductivity of the tissues, m_b the rate of blood mass flow, c_b the specific heat of the blood, S the rate of metabolic heat generation, $\bar{c}$ the specific heat of the tissue, ρ the tissue density, T_b the body core temperature, h the heat transfer coefficient, T_a the atmospheric temperature, L the latent heat of evaporation and E the evaporation rate. The terms on the left of (1) are respectively the Fickian diffusion, perfusion due to blood flow and the rate of metabolic heat generation per unit volume.

Equation (1) was used by Perl [5], Cooper and Trezek [3], [4] to study some simple problems of infinite tissue media assuming the above parameters as constants throughout the region. Chao *et al* [1] and Chao and Yang [2] investigated the steady state and unsteady state models with all parameters as constants. Saxena and coworkers [6, 7, 9, 10, 13] used analytical and numerical methods to find solutions to certain unsteady state and steady state problems of temperature distribution in skin and subcutaneous tissues. They also found [8, 12] exact solutions to steady state problems by using analytical methods.

In this paper we consider the whole region to be composed of two layers—epidermis and dermis—with skin surface insulated at time $t = 0$ and apply equations (1) and (2) for a one-dimensional unsteady state case. Denoting the thickness of epidermis and dermis by a_1 and $a_2 - a_1$ respectively, (1) is solved seperately for each layer under the following conditions:

(i) *Outer skin surface*

$$\frac{\partial \theta_1}{\partial y} = \frac{\alpha h}{K_1}\theta_1 - \left[\frac{\alpha h}{K_1}\theta_a + \frac{LE\alpha}{K_1 T_b}\right] \tag{3}$$

at $y = -a_1/\alpha$

(ii) *Epidermis*

$$K = K_1, \quad M_1 = S_1 = 0.$$

$$\partial^2 \theta_1/\partial y^2 = (\rho \bar{c}\alpha^2/K_1)(\partial \theta_1/\partial t) \tag{4}$$

(iii) *Interface*

$$K_1(\partial \theta_1/\partial y) = K_2(\partial \theta_2/\partial y) \tag{5}$$

and

$$\theta_1 = \theta_2 \quad \text{at} \quad y = 0. \tag{6}$$

(iv) *Dermis*

$$K = K_2, \quad M_2 = M[(x - a_1)/(a_2 - a_1)],$$

$$S_2 = s(T_b - T_2)[(x - a_1)/(a_2 - a_1)]$$

$$\partial^2\theta_2/\partial y^2 = [\alpha^2(M + s)/K_2]\theta_2 y = (\alpha^2\rho\bar{c}/K_2)(\partial\theta_2/\partial t) \tag{7}$$

for $0 \leqslant y \leqslant 1$.

(v) *Inner boundary*

$$\theta_2 = 0 \quad \text{at} \quad y = 1. \tag{8}$$

(vi) *Initial condition*

$$\theta_i(x, 0) = 0, \quad (i = 1, 2) \tag{9}$$

$$\text{as } T_i = T_b \quad \text{at} \quad t = 0,$$

where

$$y = [(x - a_1)/(a_2 - a_1)], \quad T_i = T_b(1 - \theta_i) \quad \text{for} \quad i = 1, 2,$$

$$T_a = T_b(1 - \theta_a), \quad M = m_b c_b, \quad \alpha = a_2 - a_1.$$

Applying Laplace transform on (3)–(8) and using initial condition (9) we have

$$\mathrm{d}\bar{\theta}_1/\mathrm{d}y = n_2\bar{\theta}_1 - n_1 \quad \text{at} \quad y = -a_1/\alpha, \tag{10}$$

$$(\mathrm{d}^2\bar{\theta}_1/\mathrm{d}y^2) - A_1\bar{\theta}_1 = 0 \quad \text{for} \quad (-a_1/\alpha) \leqslant y \leqslant 0, \tag{11}$$

$$K_1(\mathrm{d}\bar{\theta}_1/\mathrm{d}y) = K_2(\mathrm{d}\bar{\theta}_2/\mathrm{d}y) \tag{12}$$

$$\text{and } \bar{\theta}_1 = \bar{\theta}_2 \quad \text{at} \quad y = 0 \tag{13}$$

$$(\mathrm{d}^2\bar{\theta}_2/\mathrm{d}y^2) + (\bar{a}_1 y + \bar{a}_2)\bar{\theta}_2 = 0 \quad \text{for} \quad 0 \leqslant y \leqslant 1 \tag{14}$$

$$\text{and } \bar{\theta}_2 = 0 \quad \text{at} \quad y = 1, \tag{15}$$

where

$$\bar{\theta}_i = \text{Laplace transform of } \theta_i = \int_0^\infty \theta_i \exp(-pt)\,\mathrm{d}t,$$

$$A_1 = (\alpha^2\rho\bar{c}/K_1)p, \quad n_1 = (\alpha h\theta_a/pK_1) + (LE\alpha/pK_1 T_b)$$

$$n_2 = \alpha h/K_1, \quad \bar{a}_1 = -\alpha^2(M + s)/K_2, \quad \bar{a}_2 = (-\alpha^2\rho\bar{c}/K_2)p$$

3. Solution

The solution for the two layers is obtained in the following form:

$$\bar{\theta}_1 = C_1 \exp(\sqrt{A_1}\,y) + C_2 \exp(-\sqrt{A_1}\,y) \quad \text{for} \quad (-a_1/\alpha) \leqslant y \leqslant 0, \tag{16}$$

$$\bar{\theta}_2 = (\bar{a}_2 + \bar{a}_1 y)^{1/2}(\bar{a}_1)^{-1/3}[C_3 W_1(\bar{x}) + C_4 W_2(\bar{x})] \quad \text{for} \quad 0 \leqslant y \leqslant 1, \tag{17}$$

where

$$W_1(\bar{x}) = J_{1/3}(\bar{x}), \quad W_2(\bar{x}) = J_{-1/3}(\bar{x}), \quad \bar{x} = \frac{2}{3a_1}(\bar{a}_2 + \bar{a}_1 y)^{3/2}$$

$J_n(\bar{x})$ is a Bessel function of order n.

The values of the constants C_1, C_2, C_3 and C_4 are obtained by using interface and boundary conditions (10), (12), (13) and (15) and are given in Appendix 1.

Now finding the inverse Laplace transform of (16) and (17) will give $\theta_1(y,t)$ and $\theta_2(y,t)$. But it is difficult to find the exact inverse Laplace transform of (16) and (17). Since the functions involved in the numerator and denominator of (16) and (17) are convergent, we can approximate the functions involved in solutions (16) and (17) by truncating the series after taking the first few terms of the series and find the inverse Laplace transform of the approximate expression obtained.

As an example we consider a simple case for numerical calculations. We expand all the functions involved in solutions (16) and (17) in series and take only the first term in the series and neglecting the rest we get

$$\bar{\theta}_1(y,p) \approx \frac{g_1[R_1 + R_2 p]}{p[q_4 p^2 + q_5 p + q_6]} - \frac{2q_1 \alpha^2 y}{p(g_4 p + g_5)}, \tag{18}$$

$$\bar{\theta}_2(y,p) \approx \frac{g_1(1-y)}{p(g_2 p + g_3)}. \tag{19}$$

Applying the inverse Laplace transform on (18) and (19) we get

$$\theta_1(y,t) \approx g_7\left[\frac{R_1}{\beta_1\beta_2} + \frac{(\beta_1 R_2 - R_1)}{\beta_1(\beta_2 - \beta_1)}\exp(-\beta_1 t) + \frac{(R_1 - \beta_2 R_2)\exp(-\beta_2 t)}{\beta_2(\beta_2 - \beta_1)}\right]$$

$$- \frac{g_6 y}{\beta_2}[1 - \exp(-\beta_2 t)] \tag{20}$$

$$\theta_2(y,t) \approx \frac{g_1(1-y)}{g_2\beta_1}[1 - \exp(-\beta_1 t)]. \tag{21}$$

The values of the constants involved in (18), (19), (20) and (21) are given in Appendix 1.

4. Numerical results and discussion

The numerical results are obtained using the following values ([3], [4] and [8]):

$K_1 = 0{\cdot}030$ cal/cm-min. °C,

$K_2 = 0{\cdot}045$ cal/cm-min. °C,

$h = 0{\cdot}009$ cal/cm^2-min. °C,

$L = 579$ cal/g,

$T_b = 37$°C,

$\rho = 1{\cdot}05$ g/cm^2,

The values of layer thickness are taken as (i) $a_1 = 0{\cdot}2$ cm, $a_2 = 1{\cdot}1$ cm and (ii) $a_1 = 0{\cdot}4$ cm, $a_2 = 1{\cdot}1$ cm. The numerical calculations have been made for three cases of atmospheric temperatures

(i) $T_a = 15°C$, $s = 0{\cdot}0357$ cal/cm^3-min,

$M = m_b c_b = 0{\cdot}003$ cal/cm^3-min. °C, $E = 0$.

(ii) $T_a = 23°C$, $s = 0{\cdot}018$ cal/cm^3-min,

$M = m_b c_b = 0{\cdot}018$ cal/cm^3-min. °C.

$E = 0$, $0{\cdot}24 \times 10^{-3}$, $0{\cdot}48 \times 10^{-3}$ g/cm^2-min.

(iii) $T_a = 33°C$, $s = 0{\cdot}018$ cal/cm^3-min,

$M = m_b c_b = 0{\cdot}0315$ cal/cm^3-min. °C.

$E = 0{\cdot}24 \times 10^{-3}$, $0{\cdot}48 \times 10^{-3}$ and $0{\cdot}72 \times 10^{-3}$ g/cm^2-min.

Curves have been plotted for $T^{(i)}$ $(i = 0, 1)$ vs time t for the same thickness of skin and different values of atmospheric temperatures. The layer thickness of epidermis and dermis are

(i) $a_1 = 0{\cdot}2$ cm, $\alpha = 0{\cdot}9$ cm,
(ii) $a_1 = 0{\cdot}4$ cm, $\alpha = 0{\cdot}7$ cm.

In figure 1 [case(i)] for $T_a = 15°C$ and $E = 0$, it can be seen that the temperatures $T^{(i)}$ $(i = 0, 1)$ fall quickly during the first five minutes and reach steady state in about twenty minutes. In figure 2 [case(i)] for $T_a = 23°C$ and different values of E, we observe that $T^{(0)}$ and $T^{(1)}$ decrease quickly during first five minutes and reach steady state in about twenty minutes. The curves are almost parallel after about fifteen minutes. The fall in $T^{(0)}$ and $T^{(1)}$ is more for higher rates of evaporation. The fall in $T^{(0)}$ and $T^{(1)}$ at $E = 0$ in this case is less compared to the previous case (figure 1).

In figure 3 [case(ii)] for $T_a = 23°C$ and different values of E, it is observed that the fall in $T^{(0)}$ and $T^{(1)}$ is more gradual and smooth and reaches steady state in about

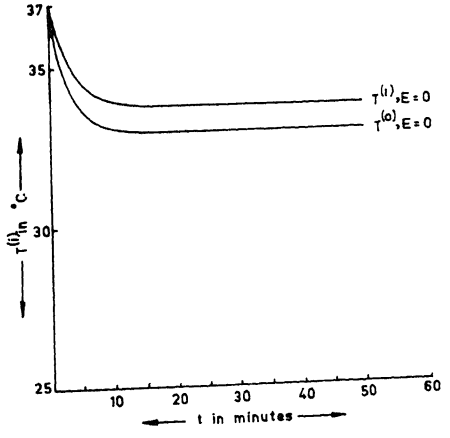

Figure 1. Plot of $T^{(i)}$ $(i = 0, 1)$ and time t for $T_a = 15°C$, $a = 0{\cdot}2$ cm, $\alpha = 0{\cdot}9$ cm and $E = 0$.

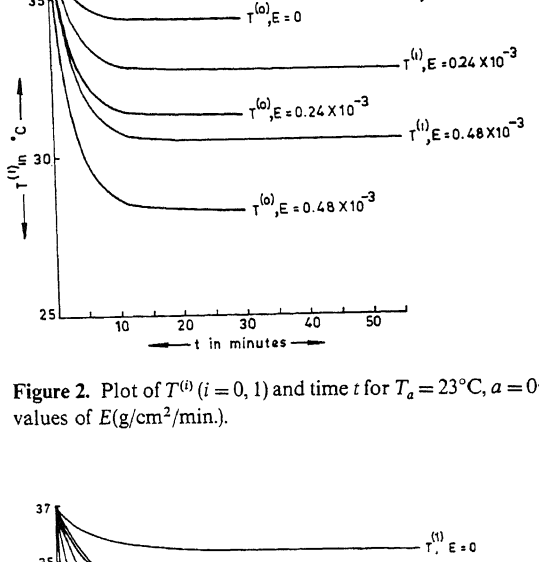

Figure 2. Plot of $T^{(i)}$ $(i = 0, 1)$ and time t for $T_a = 23°C$, $a = 0{\cdot}2$ cm, $\alpha = 0{\cdot}9$ cm and for differe values of E(g/cm^2/min.).

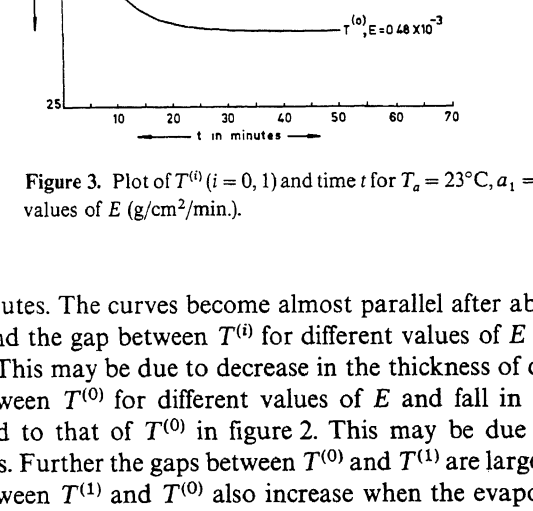

Figure 3. Plot of $T^{(i)}$ $(i = 0, 1)$ and time t for $T_a = 23°C$, $a_1 = 0{\cdot}4$ cm, $\alpha = 0{\cdot}7$ cm and for differ values of E (g/cm^2/min.).

forty minutes. The curves become almost parallel after about thirty minutes. The f in $T^{(1)}$ and the gap between $T^{(i)}$ for different values of E are less than that of $T^{(1)}$ figure 2. This may be due to decrease in the thickness of dermis in this case. Also t gaps between $T^{(0)}$ for different values of E and fall in $T^{(0)}$ in this case are larg compared to that of $T^{(0)}$ in figure 2. This may be due to increase in thickness epidermis. Further the gaps between $T^{(0)}$ and $T^{(1)}$ are larger than those in figure 2. T gaps between $T^{(1)}$ and $T^{(0)}$ also increase when the evaporation rate is increased.

In figure 4 [case(i)] for $T_a = 33°C$ and different values of E, the temperatures T and $T^{(1)}$ fall quickly during the first five minutes and reach steady state in abo twenty minutes. The curves become almost parallel after about fifteen minutes. T gaps between $T^{(0)}$ and $T^{(1)}$ for different values of E are smaller than those in figure The fall in $T^{(0)}$ and $T^{(1)}$ is also less in comparison to that in figure 2. This may due to decrease in heat loss from the skin surface by convection and radiation wh

Table 1. Thickness of SST region = 1·1 cm.

Atmospheric temperature (°C) and rate of evaporation ($\times 10^{-3}$)(g/cm^2-min.)	$T_a = 15$°C	[$T_a = 23$°C]			[$T_a = 33$°C]		
	$E = 0$	$E = 0$	$E = 0{\cdot}24$	$E = 0{\cdot}48$	$E = 0{\cdot}24$	$E = 0{\cdot}48$	$E = 0{\cdot}72$
Steady state $T^{(0)}$ Saxena and Bindra [11]	33·1733	34·5562	31·8202	29·0846	33·6037	30·8677	28·1318
Steady state $T^{(0)}$ Our results	32·962997	34·29039	31·302078	28·313751	33·237522	30·249191	27·260871
Difference in %	0·637997	0·7751734	1·65523	2·722525	1·10170066	2·04471	3·19479

Table 2. Thickness of SST region = 1·1 cm.

Atmospheric temperature (°C) and rate of evaporation ($\times 10^{-3}$)(g/cm^2-min.)	$T_a = 15$°C	$T_a = 23$°C			$T_a = 33$°C		
	$E = 0$	$E = 0$	$E = 0{\cdot}24$	$E = 0{\cdot}48$	$E = 0{\cdot}24$	$E = 0{\cdot}48$	$E = 0{\cdot}72$
Steady state $T^{(1)}$ at $x = 0{\cdot}2$ cm Saxena and Bindra [11]	34·2638	35·2497	33·2775	31·3057	34·5679	32·5958	30·6237
Steady state $T^{(1)}$ at $x = 0{\cdot}2$ cm Our results	33·809416	34·967801	32·72657	30·482779	34·178143	31·936911	29·695678
Difference in %	1·34	0·806	1·68	2·6996	1·140	2·063	3·125

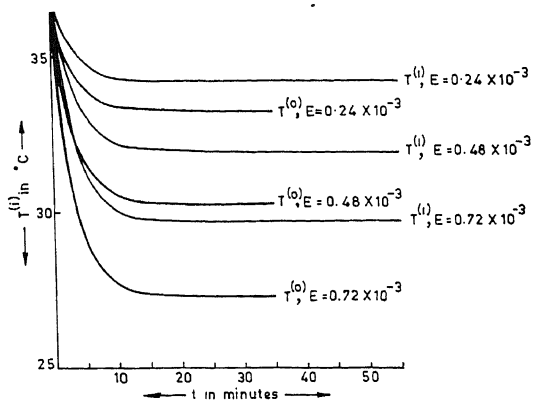

Figure 4. Plot of $T^{(i)}(i=0,1)$ and time t for $T_a = 33°C$, $a_1 = 0{\cdot}2$ cm, $\alpha = 0{\cdot}9$ cm and for different values of E (g/cm²/min.).

in turn depends on the temperature gradient at the skin surface. The fall in temperature gradient is due to increase in atmospheric temperature. Thus the rate of sweat evaporation plays a major role in heat loss at the skin surface in this case.

From the above observations we conclude that the evaporation rate plays an important role in heat loss from skin surface at higher atmospheric temperatures. The skin of same thickness but with thicker epidermis and thinner dermis layers provides better insulation to the body core. We also conclude that the dermis has a higher thermal conductivity than the epidermis.

Our results are comparable with those obtained by Saxena and Bindra [11] using the variational finite element method [see tables 1 and 2], who considered those regions which have all the three layers namely epidermis, dermis and subcutaneous tissues. The difference in the results may be because the terms involving m and s in our case got cancelled in the numerator and denominator of (16) and (17). Also we have taken those regions with no subcutaneous tissues. However, despite our limitations, the method appears quite efficient for the situations studied in the problem and gives useful results.

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Appendix 1

$$C_1 = \left[\frac{4n_3 n_4 \bar{a}_2 F_1 n_5 \sqrt{A_1}}{(n_3+1)F_3} - 1\right], \quad C_2 = \left[\frac{4n_4 \bar{a}_2 n_5 F_1 \sqrt{A_1}}{(n_3+1)F_3} + 1\right],$$

$$C_3 = \frac{4n_4(\bar{a}_2)^{1/2}(\bar{a}_1)^{1/3} n_5 \sqrt{A_1}\, l_2}{F_3}, \quad C_4 = \frac{-4n_4(\bar{a}_2)^{1/2}(\bar{a}_1)^{1/3} n_5 \sqrt{A_1}\, l_1}{F_3},$$

$$F_1 = l_2 l_4 - l_1 l_5, \quad F_2 = l_2 l_7 - l_1 l_8,$$

$$F_3 = 2\bar{a}_2 n_5 \sqrt{A_1}(n_3 - 1)F_1 - (n_3 + 1)\{\bar{a}_1 F_1 + 2\bar{a}_2 F_2\},$$

$$n_3 = \frac{\exp[2\sqrt{A_1}(a_1/\alpha)][\sqrt{A_1} + n_2]}{[\sqrt{A_1} - n_2]},$$

$$n_4 = \frac{n_1 \exp[\sqrt{A_1}(a_1/\alpha)]}{[\sqrt{A_1} - n_2]}, \quad n_5 = K_1/K_2,$$

$$l_1 = \frac{(\bar{a}_2 + \bar{a}_1)^{1/2}}{\Gamma(\frac{4}{3})(3\bar{a}_1)^{1/3}} l_9, \quad l_2 = \frac{(\bar{a}_2 + \bar{a}_1)^{-1/2}}{\Gamma(\frac{2}{3})(3\bar{a}_1)^{-1/3}} l_{10},$$

$$l_3 = {}_0F_1\left[-; 4/3 \frac{-(\bar{a}_2 + \bar{a}_1 y)^3}{9(\bar{a}_1)^2}\right], \quad l_4 = \frac{(\bar{a}_2)^{1/2}}{\Gamma(\frac{4}{3})(3\bar{a}_1)^{1/3}} l_{11},$$

$$l_5 = \frac{(\bar{a}_2)^{-1/2}}{\Gamma(\frac{2}{3})(3\bar{a}_1)^{-1/3}} l_{12}, \quad l_6 = {}_0F_1\left[-; \tfrac{2}{3}; \frac{-(\bar{a}_2 + \bar{a}_1 y)^3}{9(\bar{a}_1)^2}\right],$$

$$l_7 = \frac{(\bar{a}_2)^{-1/2}}{\Gamma(\frac{1}{3})(3\bar{a}_1)^{-2/3}} l_{13} - \frac{(\bar{a}_2)^{-1/2}}{2\Gamma(\frac{1}{3})(3)^{1/3}(\bar{a}_1)^{-2/3}} l_{11},$$

$$l_8 = \frac{3(\bar{a}_2)^{-3/2} l_{14}}{\Gamma(-\frac{1}{3})(3)^{-1/3}(\bar{a}_1)^{-4/3}} + \frac{(\bar{a}_2)^{-3/2}}{2\Gamma(\frac{2}{3})(3)^{-1/3}(\bar{a}_1)^{-4/3}} l_{12},$$

$$J_{1/3}(\bar{x}) = \frac{(\bar{a}_2 + \bar{a}_1 y)^{1/2}}{\Gamma(\frac{4}{3})(3\bar{a}_1)^{1/3}} l_3, \quad J_{-1/3}(\bar{x}) = \frac{(\bar{a}_2 + \bar{a}_1 y)^{-1/2}}{\Gamma(\frac{2}{3})(3\bar{a}_1)^{-1/3}} l_6$$

$$l_9 = {}_0F_1\left[-; \tfrac{4}{3}; \frac{-(\bar{a}_2 + \bar{a}_1)^3}{9(\bar{a}_1)^2}\right], \quad l_{10} = {}_0F_1\left[-; \tfrac{2}{3}; \frac{-(\bar{a}_2 + \bar{a}_1)^3}{9(\bar{a}_1)^2}\right]$$

$$l_{11} = {}_0F_1\left[-; \tfrac{4}{3}; \frac{-(\bar{a}_2)^3}{9(\bar{a}_1)^2}\right], \quad l_{12} = {}_0F_1\left[-; \tfrac{2}{3}; \frac{-(\bar{a}_2)^3}{9(\bar{a}_1)^2}\right]$$

$$l_{13} = {}_0F_1\left[-; \tfrac{1}{3}; \frac{-(\bar{a}_2)^3}{9(\bar{a}_1)^2}\right], \quad l_{14} = {}_0F_1\left[-; -\tfrac{1}{3}; -\frac{(\bar{a}_2)^3}{9(\bar{a}_1)^2}\right],$$

$$g_1 = 2n_5\alpha^2 q_1, \quad q_1 = (\alpha h \theta_a / K_1) + (LE\alpha / K_1 T_b).$$

$$g_2 = a_1^2(1 + n_2 n_5)q_3 + 2a_1 \alpha n_5 q_3, \quad q_3 = \alpha^2 \rho \bar{c}/K_1$$

$$g_3 = 2\alpha^2(1 + n_2 n_5) + 2a_1 \alpha n_2, \quad g_4 = q_3 a_1^2$$

$$g_5 = 2\alpha^2 + 2n_2 a_1 \alpha, \quad g_6 = 2q_1 \alpha^2 / g_4, \quad g_7 = g_1/g_2 g_4$$

$$\beta_1 = g_3/g_2, \quad \beta_2 = g_5/g_4,$$

$$R_1(y) = 2\alpha^2 + 2n_2 \alpha(a_1 + \alpha y), \quad R_2(y) = q_3(a_1 + \alpha y)^2$$

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A class of exact solutions and physical behaviour of magnetic field lines in plasma column

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Abstract. Using hodograph and Legendre transformation functions the basic equations are recast in terms of this function, and the conditions which this function should satisfy are stated. Several applications of this method are considered, and the geometry of magnetic field lines is discussed.

Keywords. Magnetic field lines; plasma column; Legendre transformation function.

1. Introduction

Geometric theories of curve and surfaces in fluid flow theories are made use of as they provide possible information on flow structures. In connection with plasma confinement, it is of interest to study the two-dimensional equilibrium in which the conducting fluid is held in place by a magnetic field. More details on this subject are contained in the encyclopaedic articles of Chandrasekhar [3], Prim and Nemenyi [11], who studied the geometry of plane flows when the vorticity is constant. Grad and Rubin [7] pointed out that one may search for hydrostatic equilibria by exploring the analogy between steady flow of an incompressible fluid and static equilibria of a conducting fluid in a magnetic field. An excellent survey of this method with application to numerous problems has been given by Ferraro and Plumpton [5]. This result was later extended by Shivamoggi [12], Singh and Tripathi [14]. In the present paper, using hodograph and Legendre transformation approach, the geometry of magnetic field lines is discussed.

2. Basic equations

The basic equations governing the magneto-hydrostatic equilibrium of a perfectly conducting fluid is characterized by [12].

$$\nabla p = \bar{\Omega} \times \bar{H}, \tag{1}$$

$$\nabla \times \bar{H} = \bar{\Omega}, \tag{2}$$

$$\nabla \cdot \bar{H} = 0, \tag{3}$$

where p is the pressure, $\bar{H}$ the magnetic field strength and $\bar{\Omega}$ the current density. Even if the electrical conductivity is finite the theory developed by taking the electrical conductivity to be infinite will still be valid for time that are short compared with the decay time of the magnetic field. From (1)

$$\bar{\Omega}\cdot\nabla p = 0, \quad \bar{H}\cdot\nabla p = 0 \tag{4}$$

i.e. the magnetic field lines and current lines must lie on the same surface $p = \text{const.}$ Equation (1) implies [10, 16, 17]

$$\text{curl}\,(\bar{\Omega}\times\bar{H}) = \bar{0} \tag{5}$$

which is the compatibility condition for any hydrostatic equilibria. Equation (5) asserts the existence of a family of surface $\phi = \text{constant}$ given by

$$\bar{\Omega}\times\bar{H} = \nabla\phi.$$

The surface $\phi = \text{constant}$ is called the *Lorentz surface.*

3. Analysis for plane case

Considering the plane case where $H = H(x, y) = H_1(x, y), H_2(x, y)$ in the physical (x, y) plane H_1 and H_2 satisfies

$$(\partial H_1/\partial x) + (\partial H_2/\partial y) = 0, \tag{6}$$

$$H_1(\partial\Omega/\partial x) + H_2(\partial\Omega/\partial y) = 0, \quad \Omega = (\partial H_2/\partial x) - (\partial H_1/\partial y), \tag{7}$$

$$H_1^2(\partial H/\partial x) + H_2^2(\partial H_2/\partial y) + H_1 H_2\left(\frac{\partial H_2}{\partial x} + \frac{\partial H_1}{\partial y}\right) = 0. \tag{8}$$

We define the Jacobian for transformation of the above equations from the physical plane to hodograph plane as $J = \partial(H_1, H_2)/\partial(x, y)$.

We shall now consider the following cases: (i) $J \neq 0$ in the entire region of the flow. (ii) $J = 0$ in the entire region of flow and (iii) $J = 0$ in a part of the region and $J \neq 0$ in the remaining region.

(i) *Case for* $J \neq 0$.

In this case the system of equations (6)–(8) can be transformed into an equivalent linear system by interchanging the roles of dependent and independent variables. For a solution $H_1(x, y)$, $H_2(x, y)$, we may consider x and y as function of H_1 and H_2. Employing the transformations

$$\partial H_1/\partial x = J(\partial y/\partial H_2), \quad \partial H_1/\partial y = -J(\partial x/\partial H_2)$$

$$\partial H_2/\partial x = -J(\partial y/\partial H_1), \quad \partial H_2/\partial y = J(\partial x/\partial H_1)$$

in (6)–(8) we see that $x(H_1, H_2)$ and $y(H_1, H_2)$ satisfy the following linear differential

equations

$$(\partial x/\partial H_1) + (\partial y/\partial H_2) = 0, \tag{9}$$

$$H_1 \frac{\partial(\Omega, y)}{\partial(H_1, H_2)} + H_2 \frac{\partial(x, \Omega)}{\partial(H_1, H_2)} = 0, \tag{10}$$

$$H_1^2 \frac{\partial y}{\partial H_2} + H_2^2 \frac{\partial x}{\partial H_1} - H_1 H_2 \left(\frac{\partial y}{\partial H_1} + \frac{\partial x}{\partial H_2} \right) = 0. \tag{11}$$

Equation (9) implies the existence of a Legendre transform function $L(H_1, H_2)$ such that

$$\mathrm{d}L = x\mathrm{d}H_2 - y\mathrm{d}H_1, \quad L_{H_1} = \partial L/\partial H_1 = -y, \quad L_{H_2} = \partial L/\partial H_2 = x. \tag{12}$$

Employing (12) in (9)–(11), (9) is identically satisfied and (10) and (11) take the form

$$H_2 \frac{\partial(L_{H_2}, \Omega)}{\partial(H_1, H_2)} - H_1 \frac{\partial(\Omega, L_{H_1})}{\partial(H_1, H_2)} = 0, \tag{13}$$

$$L_{H_1 H_2}(H_2^2 - H_1^2) - (L_{H_1 H_1} - L_{H_2 H_2}) H_1 H_2 = 0, \tag{14}$$

where

$$\Omega = J(L_{H_1 H_2} - L_{H_2 H_2}),$$

$$J = [L_{H_1 H_1} L_{H_2 H_2} - (L_{H_1 H_2})^2]^{-1}.$$

Defining polar coordinates (H, θ) in the (H_1, H_2) plane given by $H^2 = (H_1^2 + H_2^2)^{1/2}$ and $\theta = \tan^{-1}(H_2/H_1)$, the above equations are reduced to

$$\frac{\partial \Omega}{\partial \theta} \frac{\partial^2 L}{\partial q^2} - \frac{\partial \Omega}{\partial q} \left(\frac{\partial^2 L}{\partial H \partial \theta} - \frac{1}{H} \frac{\partial L}{\partial \theta} \right) = 0, \tag{15}$$

$$\frac{\partial^2 L}{\partial \theta \partial H} - \frac{1}{H} \frac{\partial L}{\partial \theta} = 0, \tag{16}$$

where

$$\Omega = J \left[H^4 \left(\frac{\partial^2 L}{\partial H^2} + \frac{1}{H^2} \frac{\partial^2 L}{\partial \theta^2} + \frac{1}{H} \frac{\partial L}{\partial H} \right) \right]$$

$$\times \left[H^2 \frac{\partial^2 L}{\partial H^2} \left(H \frac{\partial L}{\partial H} + \frac{\partial^2 L}{\partial \theta^2} \right) - \left(\frac{\partial L}{\partial \theta} - H \frac{\partial^2 L}{\partial H \partial \theta} \right)^2 \right]^{-1}. \tag{17}$$

The general solution of (16) is found to be

$$L(H, \theta) = H\phi(\theta) + \chi(H), \tag{18}$$

where $\phi(\theta)$ and $\chi(H)$ are the arbitrary functions of their arguments. From (18) and (15), we have

$$\frac{\partial}{\partial \theta} \left[\frac{\phi(\theta) + \phi''(\theta) + H\chi''(H) + \chi'(H)}{\{\chi(H)\phi(\theta) + \phi''(\theta) + \chi'(H)\}} \right] \chi''(H) = 0, \tag{19}$$

Simplifying (19), we find that $\phi(\theta)$ satisfies

$$\phi(\theta)+\phi''(\theta)=0. \tag{20}$$

The general solution of (19) is given by

$$\phi(\theta)=C_1+C_2\cos\theta+C_3\sin\theta, \tag{21}$$

where C_i^s are arbitrary constants. Hence from (21) and (18) we have

$$L(H,\theta)=H(C_1+C_2\cos\theta+\sin\theta)+\chi(H). \tag{22}$$

Using this solution in (12), in (H,θ) coordinates, we have

$$\begin{aligned} x-C_3&=[C_1+\chi'(H)]\sin\theta\\ y+C_2&=[C_1+\chi'(H)]\cos\theta. \end{aligned} \tag{23}$$

Squaring and adding we obtain

$$(x-C_3)^2+(y+C_2)^2=[C_1+\chi'(H)]^2. \tag{24}$$

From (24), we observe that the *magnetic field lines are concentric circles* with centre $(+C_3-C_2)$.

Defining the distance of any point (x,y) from the centre (C_3,C_2) by R, we obtain from (24) that $R=C_1+\chi'(H)$. Differentiating this result with respect to H, we obtain $dR/dH=\chi''(H)$. From (19) it is required that $\chi''(H)\neq 0$, and therefore we have the previous result $H=H(R)$.

Using this result in (23), the field line components are given by

$$\begin{aligned} H_1&=-\frac{(y+C_2)H(R)}{\chi'[H(R)+C_1]}=-\frac{(y+C_2)H(R)}{R},\\ H_2&=\frac{(x-C_3)H(R)}{\chi'[H(R)+C_1]}=\frac{(x-C_3)H(R)}{R}. \end{aligned} \tag{25}$$

(ii) *Case* (2) $(J=0)$

The Jacobian being zero implies that

$$H_2=\phi(H_1), \tag{26}$$

where ϕ is an arbitrary function of H_1. Using (26) in (8) we find that $H_1(x,y)$ must satisfy

$$[H_1+\phi(H_1)\phi'(H_1)]\left[H_1\frac{\partial H_1}{\partial x}+\phi'(H_1)\frac{\partial H_1}{\partial y}\right]=0. \tag{27}$$

From the above it follows that either the first or the second term or both the terms are zero. This, in conjunction with (6) and (7), gives rise to the following system of equations as

$$H_1(\partial H_1/\partial x)+\phi(H)(\partial H_1/\partial y)=0,$$

$$(\partial H_1/\partial x) + (\partial H_2/\partial y) = 0, \tag{28}$$

$$H_1(\partial\Omega/\partial x) + H_2(\partial\Omega/\partial y) = 0,$$

$$H_1 + \phi(H_1)\phi'(H_1) = 0,$$

$$(\partial H_1/\partial x) + (\partial H_2/\partial y) = 0, \tag{29}$$

$$H_1(\partial\Omega/\partial x) + H_2(\partial\Omega/\partial y) = 0.$$

For a magnetic field given by $H_1(x, y)$, $H_2(x, y) = \phi(H_1)$ any one of the above system of equations must be satisfied.

Using the first two equations of the system of (28) we observe that

$$[H_1\phi'(H_1) - \phi(H_1)](\partial H_1/\partial y) = 0. \tag{30}$$

Equation (30) is satisfied either by putting

$$\partial H_1/\partial y = 0 \quad \text{or} \quad H_1\phi'(H_1) - \phi(H_1) = 0.$$

The first possibility leads to $\partial H_1/\partial y = 0$, and therefore $H_1 = H_0$ and $H_2 = \phi(H_1) = H_0^*$, where H_0 and H are two arbitrary real numbers. Hence the current density is zero and field lines are parallel straight lines. On integrating, the second possibility leads to $\phi(H_1) = k_1 H_1$. Substituting this result in the second equation of this system, we find that $\phi H_1(x, y)$ satisfies

$$(\partial H_1/\partial x) + (\partial H_1/\partial y) = 0. \tag{31}$$

Therefore

$$H_1(x, y) = f(k_1 x - y)$$

and

$$H_2(x, y) = k_1 H_1 = k_1 f(k_1 x - y),$$

where f is an arbitrary function. Using the above result, the current density Ω is given by

$$\Omega = (k_1^2 + 1) f(k_1 x - y). \tag{32}$$

The third equation of the system is identically satisfied by H_1, H_2 and Ω.

Considering (29) and integrating the first equation with respect to H_1, we obtain

$$H_1^2 + H_2^2 = k_2, \tag{33}$$

where $k_2 > 0$ is an arbitrary constant.

Introducing the magnetic flux function $\phi(x, y)$, which is given by $d\phi = H_1\, dy - H_2\, dx$; the second equation of this system is identically satisfied and (33) becomes

$$[(\partial\phi/\partial y)^2 + (\partial\phi/\partial x)^2] = k_2. \tag{34}$$

The complete integral of (34) is given by

$$\phi(x, y) = (k_2)^{1/2}[x\cos\alpha + y\sin\alpha + \beta(\alpha)], \tag{35}$$

where α and β are two arbitrary parameters.

The general solution of (34) is obtained by taking $\beta = \beta(\alpha)$ and eliminating α between

$$\phi = k_2^{1/2}[x\cos\alpha + y\sin\alpha + \beta(\alpha)], \tag{36}$$

$$0 = -x\sin\alpha + y\cos\alpha + \beta(\alpha). \tag{37}$$

Differentiating (36) with respect to x and y and using (37) we have

$$\partial\phi/\partial x = p = k_2^{1/2}\cos\alpha, \quad \partial\phi/\partial y = g = k_2^{1/2}\sin\alpha. \tag{38}$$

Again differentiating (38) with respect to x and y we have

$$\partial^2\phi/\partial x^2 = r = -k_2^{1/2}\sin\alpha(\partial\alpha/\partial x), \tag{39}$$

$$\partial^2\phi/\partial y^2 = t = k_2^{1/2}\cos\alpha(\partial\alpha/\partial y), \tag{40}$$

$$\partial^2\phi/\partial x\,\partial y = -k_2^{1/2}\sin\alpha(\partial\alpha/\partial y) = k_2^{1/2}\cos\alpha(\partial\alpha/\partial x). \tag{41}$$

Using (38)–(41) in (7) we obtain

$$\sin\alpha(\partial/\partial\alpha)[-\sin\alpha(\partial\alpha/\partial x) + \cos\alpha(\partial\alpha/\partial y)].$$

Employing (41) in this relation, we obtain

$$(\partial^2\alpha/\partial x^2) + (\partial^2\alpha/\partial y^2) = 0. \tag{42}$$

Differentiating (37) twice with respect to x and twice with respect to y and then adding we get

$$[\beta(\alpha) + \beta''(\alpha)][(\partial\alpha/\partial x)^2 + (\partial\alpha/\partial y)^2] = 0. \tag{43}$$

Equation (43) implies that either the first term or the second term is zero.

The first possibility gives $\alpha =$ constant and, therefore, the field lines are a family of parallel straight lines. The second possibility gives

$$\beta(\alpha) = C_4 + C_5\sin\alpha - C_6\cos\alpha, \tag{44}$$

where C_4's are the arbitrary constants. Using this expression for $\beta(\alpha)$ in (36) we have

$$[(1/\sqrt{k_2}) - C_1] = (x - C_3)\cos\alpha + (y + C_2)\sin\alpha \tag{45}$$

$$0 = -(x - C_3)\sin\alpha + (y - C_2)\cos\alpha. \tag{46}$$

Squaring and adding, we eliminate

$$[(1/\sqrt{k_2}) - C_1] = \pm[(x - C_3)^2 - (y - C_2)^2]^{1/2} \tag{47}$$

which implies that the magnetic field lines are concentric circles with

$$H_1 = \pm\left[\frac{k_2(y - C_2)}{(x - C_3)^2 + (y - C_2)^2}\right]^{1/2}, \tag{48}$$

$$H_2 = \pm\left[\frac{k_2(x - C_3)}{(x - C_3)^2 + (y - C_2)^2}\right]^{1/2}.$$

(iii) *Case* (3) ($J = 0$) in a part of the region and $J \neq 0$ in the remaining parts of the region.

From the above analysis it is apparent that the common magnetic field line patterns for the case $J = 0$, and $J \neq 0$ are concentric circles if we restrict our attention to any one magnetic field line, a part of which lies in each region. We find from (25) and (32) that we encounter as we cross from one region to another. Therefore, such regions cannot exist.

4. Applications

In this section, information about the hodograph and Legendre technique and its uses to develop solution with lower symmetrics (ellipses, parabolas etc) with examples are given. Hamilton, in 1846, coined the word 'hodograph' for a velocity locus with a moving particle. If the hodograph field equations are easier to solve (subject to the boundary conditions as may be prescribed) then the original equations the hodograph method will be advantageous. One such important case where this occurs is that of a steady and ideal, inviscid gas in two dimensions whose Eulerian equation is

$$(c^2 - u^2)u_x - uv(u_y + v_x) + (c^2 - v^2)v_y = 0, \quad u_y - v_x = 0, \tag{49}$$

where c the velocity of sound at the point (x, y) in the flow field is nonlinear but whose hodograph equations

$$(c^2 - u^2)y_v + uv(x_v + y_u) + (c^2 - v^2)x_u = 0, \quad x_v - y_u = 0 \tag{50}$$

are linear. The hodograph relations for stream function are

$$\psi_u = (1 - q^2)^\beta (vx_u - uy_v), \quad \psi_v = (1 - q^2)^\beta (vx_v - uy_v), \tag{51}$$

where $q^2 = u^2 + v^2$, $\beta = 1/(r - 1)$, $q^2 + 2\beta c^2 = 1$ and $p = k\rho^r$ (r is the ratio of specific heats).

A solution of (50) is a pair of relations

$$x = x(u, v), \quad y = y(u, v),$$

whose inversion u, v must satisfy (49). In practice we can sometimes find explicit analytic solutions for (50) and (51) which is practicable because these equations are linear and relatively simple in form. The hodograph procedure enables us to obtain the configurations with uniform accuracy, out to a natural boundary of the flow field. This would not be practicable by direct numerical solutions of the nonlinear equation (49).

Special solutions of nonlinear problems have been obtained by numerous authors [1, 2, 6, 8, 9, 15] using hodograph or modifications. For our simple example we recall (49) for the stream function ψ. Every solution $\psi(u, v)$ defined in a certain region of u, v plane leads to a flow given by u and v as functions of x and y provided the Jacobian

$$J = x_u y_v - x_v y_u = \psi_{uu}\psi_{vv} - \psi_{uv^2}$$

is not zero. Equation (50) is elliptic for subsonic and hyperbolic for supersonic flow. For subsonic flow J never vanishes unless $\psi_{uu} = \psi_{vv} = \psi_{uv} = 0$. The advantage of

linearity gained by this hodograph is paid for by boundary condition complication. On the other hand, the problem of supersonic flow is complicated as J changes its sign, in which case, the functions x, y do not represent a flow globally since the image in (x, y) plane possesses a fold covering part of the plane two or three times. Despite such singularities, special significant flow field is obtained by hodographic means for supersonic and subsonic flows. A lengthy discussion of the hodograph and its singularities are found in Courant and Friedrichs [4, pp. 62 and 256], using hodograph and Legendre transform technique. Shivamoggi [15, pp. 218] showed that for a two-dimensional steady potential flow

$$y^2 = k^2[(2x/R^2) - 1], \quad k = \text{constant}$$

which represents a family of confocal parabolic symmetric curves about x-axis which focus at origin.

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Propagation of waves in a micropolar elastic layer with stretch immersed in an infinite liquid

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Abstract. The effect of liquid on the propagation of waves in a micropolar elastic layer with stretch has been investigated. The frequency and wave velocity equations for symmetric and antisymmetric vibrations are derived. Propagation of monochromatic waves in a micropolar elastic layer with stretch is discussed. Results of this analysis reduce to those without stretch.

Keywords. Wave propagation; micropolar elastic layer; monochromatic waves; symmetric vibrations; antisymmetric vibrations.

1. Introduction

Modern engineering structures are often made up of materials possessing an internal structure. Polycrystalline materials, materials with fibrous or coarse grain structure come in this category. Classical elasticity is inadequate to represent the behaviour of such materials. For the analysis of such materials, there is need to incorporate the theory of oriented media. Describing the behaviour of oriented media, the micropolar theory is one of these theories.

The basic equations of linear micropolar theory of elasticity have been given by Kuvchinski and Aero [5], Palmov [7], Eringen and Suhubi [2]. Eringen [1] extended his work to include the effect of axial stretch during the rotation of molecules and developed the theory of micropolar elastic solids with stretch. These solids respond to intrinsic rotational motions and spin inertia and therefore can support couple stresses and body couples. In addition, dipolar microelement of such solids can undergo axial intrinsic motions. The model introduced is conjectured to explain the motions of certain classes of granular and composite materials in which grains and fibres are elastic in the direction of their main axes.

The mechanical model underlying the theory of micropolar elastic solids with stretch can be envisioned as an elastic medium composed of a large number of short springs. These springs possess average inertia moments and can deform in axial direction. Thus material particles in this continuum possess not only classical translation degrees of freedom represented by the deformation vector field, but also intrinsic rotations and intrinsic axial stretch.

Due to theoretical and practical importance, many problems of waves and vibrations of micropolar elasticity have been investigated by Sengupta and De [9] and Rao and Rao [8]. Propagation of monochromatic waves in an infinite micropolar elastic plate

has been studied by Nowacki [6]. Sengupta and Benerji [10] discussed propagation of waves in a micropolar elastic layer immersed in an infinite liquid. Kumar and Chadha [3] and Kumar *et al* [4] investigated Lamb's plane problem in a micropolar elastic medium with stretch and a micropolar thermoelastic medium with stretch respectively.

In the present investigation, we discuss propagation of waves in a micropolar elastic layer with stretch immersed in an infinite liquid.

2. Formulation of the problem

We consider a micropolar elastic, homogeneous, isotropic and controsymmetric body which can stretch. We also consider a micropolar elastic layer with stretch being placed between two semi-infinite liquid media having the same properties. Let monochromatic waves propagate along the x-axis of a micropolar elastic layer with stretch of thickness $2H$ and immersed in an infinite liquid. Let the boundary of the layer be given by $z = \pm H$ and z-axis be directed downwards.

Following Eringen [1] and Nowocki [6], the field equations of micropolar elastic solids with stretch, in the absence of body forces and body couples, can be written as

$$\begin{aligned}
&(\mu+\alpha)\nabla^2\mathbf{u}+(\lambda+\mu-\alpha)\,\text{grad div}\,\mathbf{u}+2\alpha\,\text{rot}\,\boldsymbol{\omega}=\rho(\partial^2\mathbf{u}/\partial t^2),\\
&(\gamma+\varepsilon)\nabla^2\boldsymbol{\omega}+(\gamma+\beta-\varepsilon)\,\text{grad div}\,\boldsymbol{\omega}+2\alpha\,\text{rot}\,\mathbf{u}-4\alpha\boldsymbol{\omega}=J(\partial^2\boldsymbol{\omega}/\partial t^2),\\
&\alpha_0\nabla^2\Phi^*-\eta_0\Phi^*=\frac{J}{2}\frac{\partial^2\Phi^*}{\partial t^2},
\end{aligned}\tag{1}$$

where $\lambda, \mu, \gamma, \alpha, \beta, \varepsilon, \alpha_0, \eta_0$ are the material constants, ρ the density, J the rotational inertia and Φ^* the scalar microstretch. Taking $\mathbf{u}=(u_1, 0, u_3)$ and $\boldsymbol{\omega}=(0, \omega_2, 0)$, the system of (1) reduces to

$$\begin{aligned}
&(\mu+\alpha)\nabla^2u_1+(\lambda+\mu-\alpha)\frac{\partial e}{\partial x}-2\alpha\frac{\partial\omega_2}{\partial z}=\rho\frac{\partial^2u_1}{\partial t^2},\\
&(\mu+\alpha)\nabla^2u_3+(\lambda+\mu-\alpha)\frac{\partial e}{\partial z}+2\alpha\frac{\partial\omega_2}{\partial x}=\rho\frac{\partial^2u_3}{\partial t^2},\\
&[(\gamma+\varepsilon)\nabla^2-4\alpha]\omega_2+2\alpha\left(\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x}\right)=J\frac{\partial^2\omega_2}{\partial t^2},\\
&\alpha_0\nabla^2\Phi^*-\eta_0\Phi^*=\frac{J}{2}\frac{\partial^2\Phi^*}{\partial t^2},
\end{aligned}\tag{2}$$

where

$$e=(\partial u_1/\partial x)+(\partial u_3/\partial z),\quad \nabla^2=(\partial^2/\partial x^2)+(\partial^2/\partial z^2).\tag{3}$$

The displacement components u_1 and u_3 may be written in terms of potential functions as

$$\begin{aligned}
u_1&=(\partial\Phi/\partial x)-(\partial\Psi/\partial z),\\
u_3&=(\partial\Phi/\partial z)+(\partial\Psi/\partial x).
\end{aligned}\tag{4}$$

Similarly for liquid

$$u_1^{(0)} = \partial\phi_0/\partial x, \quad u_3^{(0)} = \partial\phi_0/\partial z,$$
$$u_1^{(2)} = \partial\phi_2/\partial x, \quad u_3^{(2)} = \partial\phi_2/\partial z, \tag{5}$$

where the superscripts (0), (2) and the displacement potentials ϕ_0, ϕ_2 are respectively for the liquids above and below the layer. Substituting (4) in (2), we obtain

$$\left(\nabla^2 - \frac{1}{C_1^2}\frac{\partial^2}{\partial t^2}\right)\Phi = 0. \tag{6}$$

$$\left(\nabla^2 - \frac{1}{C_2^2}\frac{\partial^2}{\partial t^2}\right)\Psi + p\omega_2 = 0, \tag{7}$$

$$\left(\nabla^2 - v_0^2 - \frac{1}{C_4^2}\frac{\partial^2}{\partial t^2}\right)\omega_2 - q\nabla^2\Psi = 0, \tag{8}$$

$$\left[\nabla^2 - \frac{1}{C_5^2}\frac{\partial^2}{\partial t^2} - v_1^2\right]\Phi^* = 0, \tag{9}$$

where

$$C_1^2 = (\lambda + 2\mu)/\rho, \quad C_2^2 = (\mu + \alpha)/\rho, \quad C_4^2 = (\gamma + \varepsilon)/J,$$
$$C_5^2 = 2\alpha_0/J, \quad p = 2\alpha/(\mu + \alpha), \quad q = 2\alpha/(\gamma + \varepsilon), \quad v_0^2 = 4\alpha/(\gamma + \varepsilon),$$
$$v_1^2 = \eta_0/\alpha_0. \tag{10}$$

Similarly for the liquid half spaces

$$\left(\nabla^2 - \frac{1}{C_\omega^2}\frac{\partial^2}{\partial t^2}\right)\phi_0 = 0, \quad \left(\nabla^2 - \frac{1}{C_\omega^2}\frac{\partial^2}{\partial t^2}\right)\phi_2 = 0, \tag{11}$$

where C_ω is the velocity of dilatational (sound) wave in the liquid.

If we eliminate Ψ or ω_2 from (7) and (8), we get

$$\left[\left(\nabla^2 - \frac{1}{C_2^2}\frac{\partial^2}{\partial t^2}\right)\left(\nabla^2 - v_0^2 - \frac{1}{C_4^2}\frac{\partial^2}{\partial t^2}\right) + \zeta^2\nabla^2\right](\psi, \omega_2) = 0, \tag{12}$$

where $\zeta^2 = pq$. To obtain the solutions of (6) and (12), we put

$$\Phi = f(z)\exp[i(kx - \omega t)],$$
$$\Psi = g(z)\exp[i(kx - \omega t)], \tag{13}$$
$$\omega_2 = h(z)\exp[i(kx - \omega t)].$$

Substituting (13) in (6) and (12) we get

$$\Phi = (A\sinh\delta z + B\cosh\delta z)\exp[i(kx - \omega t)] \tag{14}$$

$$\Psi = [C\sinh\lambda_1 z + D\cosh\lambda_1 z + E\sinh\lambda_2 z + F\cosh\lambda_2 z]$$
$$\times \exp[i(kx - \omega t)], \tag{15}$$

$$\omega_2 = [C_1 \sinh \lambda_1 z + D_1 \cosh \lambda_1 z + E_1 \sinh \lambda_2 z + F_1 \cosh \lambda_2 z] \times \exp[i(kx - \omega t)] \tag{16}$$

$$\Phi^* = [M \sinh \lambda_5 z + N \cosh \lambda_5 z] \exp[i(kx - \omega t)] \tag{17}$$

where $\delta^2 = k^2 - (\omega^2/C_1^2)$.

$$\lambda_1^2, \lambda_2^2 = k^2 + \frac{1}{2}\left[\left(v_0^2 - \zeta^2 - \frac{\omega^2}{C_2^2} - \frac{\omega^2}{C_4^2}\right) \pm \left\{\left(v_0^2 - \zeta^2 - \frac{\omega^2}{C_2^2} - \frac{\omega^2}{C_4^2}\right)^2 - 4\frac{\omega^2}{C_2^2}\left(\frac{\omega^2}{C_4^2} - v_0^2\right)\right\}^{\frac{1}{2}}\right]$$

$$\lambda_5^2 = k^2 + v_1^2 - (\omega^2/C_5^2), \tag{18}$$

$$C_1 = aC, \quad D_1 = aD, \quad E_1 = bE, \quad F_1 = bF \quad \text{and}$$

$$a = \left(k^2 - \lambda_1^2 - \frac{\omega^2}{C_2^2}\right)\Big/ p, \quad b = \left(k^2 - \lambda_2^2 - \frac{\omega^2}{C_2^2}\right)\Big/ p .$$

Again from (11), we obtain

$$\left.\begin{aligned} \phi_0 &= G \exp[i(kx - \omega t - i\lambda_3 z)], \\ \phi_2 &= P \exp[i(kx - \omega t + i\lambda_3 z)], \end{aligned}\right\} \tag{19}$$

where

$$\lambda_3^2 = k^2 - (\omega^2/C_\omega^2). \tag{20}$$

The solutions (19) have been written in such a form that the radiation conditions are satisfied.

3. Boundary conditions

Following Eringen [1] and Nowacki [6], the stress and stress moments are given by

$$\begin{aligned} t_{ji} &= \lambda u_{k,k}\delta_{ji} + (\mu - \alpha)(u_{i,j} + u_{j,i}) + 2\alpha(u_{i,j} - \varepsilon_{kji}\omega_k), \\ m_{ji} &= \beta_0 \varepsilon_{kji}\Phi^*_{,k} + \beta\omega_{k,k}\delta_{ji} + (\gamma + \varepsilon)\omega_{i,j} + (\gamma - \varepsilon)\omega_{j,i} \\ \beta_j &= \alpha_0 \Phi^*_{,j} + \frac{\beta_0}{3}\varepsilon_{kji}\omega_{k,i}. \end{aligned} \tag{21}$$

In terms of Φ and Ψ

$$\begin{aligned} t_{33} &= \lambda\nabla^2\Phi + 2\mu\left(\frac{\partial^2\Phi}{\partial z^2} + \frac{\partial^2\Psi}{\partial x\,\partial z}\right), \\ t_{31} &= \mu\left[2\frac{\partial^2\Phi}{\partial x\,\partial z} - \frac{\partial^2\Psi}{\partial z^2} + \frac{\partial^2\Psi}{\partial x^2}\right] - \alpha(\nabla^2\Psi + 2\omega_2), \\ m_{32} &= (\gamma + \varepsilon)\frac{\partial\omega_2}{\partial z} - \beta_0\frac{\partial\Phi^*}{\partial x}, \\ \beta_3 &= \alpha_0\frac{\partial\Phi^*}{\partial z} + \frac{\beta_0}{3}\frac{\partial\omega_2}{\partial x}. \end{aligned} \tag{22}$$

For the liquid

$$t_{33} = -\rho_\omega \omega^2 \phi_0 \quad \text{or} \quad -\rho_\omega \omega^2 \phi_2, \tag{23}$$

accordingly, as the liquid is above or below the layer and ρ_ω is the density of the liquid.

The boundary conditions are the continuity of the normal displacement components and the normal stress at $z = \pm H$ and the vanishing of tangential stress, couple stress and vector first moment at $z = \pm H$

$$\begin{aligned} &t_{33} \text{ for the layer} = t_{33} \text{ for the liquid} \\ &u_3 \text{ for the layer} = u_3 \text{ for the liquid} \\ &t_{31} = m_{32} = \beta_3 = 0 \text{ for the layer.} \end{aligned} \tag{24}$$

We shall now discuss two problems, one by considering the symmetric vibrations and the other by considering antisymmetric vibrations.

4. Symmetric vibrations

Considering the symmetry of the displacement u_1, the force stress t_{11} and t_{33}, the couple stress m_{32} and the first moment β_3 with respect to the plane $z = 0$, we have from (14)–(17),

$$\Phi = B \cosh(\delta z) \exp[i(kx - \omega t)], \tag{25}$$

$$\Psi = [C \sinh(\lambda_1 z) + E \sinh(\lambda_2 z)] \exp[i(kx - \omega t)], \tag{26}$$

$$\omega_2 = [C_1 \sinh(\lambda_1 z) + E_1 \sinh(\lambda_2 z)] \exp[i(kx - \omega t)], \tag{27}$$

$$\Phi^* = N \cosh(\lambda_5 z) \exp[i(kx - \omega t)]. \tag{28}$$

Substituting the values of $\Phi, \Psi, \omega_2, \Phi^*$ from (25)–(28), ϕ_0 and ϕ_2 from (19) in the boundary conditions (24), using (22), (23), (4) and (5) and taking $G = P$, we get

$$d \cosh(\delta H)B + 2i\mu k\lambda_1 \cosh(\lambda_1 H)C + 2i\mu k\lambda_2 \cosh(\lambda_2 H)E + \rho_\omega \omega^2 \exp(-\lambda_3 H)G = 0, \tag{29}$$

$$2i\mu k\delta \sinh(\delta H)B - d_1 \sinh(\lambda_1 H)C - d_2 \sinh(\lambda_2 H)E = 0, \tag{30}$$

$$(\gamma + \varepsilon)\lambda_1 a \cosh(\lambda_1 H)C + (\gamma + \varepsilon)\lambda_2 b \cosh(\lambda_2 H)E - i\beta_0 k \cosh(\lambda_5 H)N = 0, \tag{31}$$

$$\delta \sinh(\delta H)B + ik \sinh(\lambda_1 H)C + ik \sinh(\lambda_2 H)E + \lambda_3 \exp(-\lambda_3 H)G = 0 \tag{32}$$

$$i\beta_0 ka \sinh(\lambda_1 H)C + i\beta_0 kb \sinh(\lambda_2 H)E + 3\alpha_0 \lambda_5 \sinh(\lambda_5 H)N = 0, \tag{33}$$

where

$$\begin{aligned} &d = (\lambda + 2\mu)\delta^2 - \lambda k^2, \quad d_1 = (\mu + \alpha)\lambda_1^2 + (\mu - \alpha)k^2 + 2\alpha a \\ &d_2 = (\mu + \alpha)\lambda_2^2 + (\mu - \alpha)k^2 + 2\alpha b. \end{aligned} \tag{34}$$

Eliminating the constants B, C, E, G and N from equations (29) to (33), we obtain

the period equations as

$$d(\gamma+\varepsilon)\left[bd_1\lambda_2 - ad_2\lambda_1\frac{\tanh(\lambda_2 H)}{\tanh(\lambda_1 H)}\right] - 4\mu^2k^2(\gamma+\varepsilon)$$
$$\times(b-a)\delta\lambda_1\lambda_2\frac{\tanh(\delta H)}{\tanh(\lambda_1 H)} + (\gamma+\varepsilon)d_3$$
$$\times\left[b\varepsilon_1\lambda_2 - a\varepsilon_2\lambda_1\frac{\tanh(\lambda_2 H)}{\tanh(\lambda_1 H)}\right]\tanh(\delta H) - 4\mu^2k^2\delta\varepsilon_3$$
$$\times\left[a\lambda_2 - b\lambda_1\frac{\tanh(\lambda_2 H)}{\tanh(\lambda_1 H)}\right]\frac{\tanh(\delta H)}{\tanh(\lambda_5 H)}$$
$$+\varepsilon_3 d_3(a\varepsilon_2 - b\varepsilon_1)\frac{\tanh(\lambda_2 H)}{\tanh(\lambda_5 H)}\tanh(\delta H)$$
$$-d\varepsilon_3(bd_1 - ad_2)\frac{\tanh(\lambda_2 H)}{\tanh(\lambda_5 H)} = 0, \tag{35}$$

where

$$\varepsilon_1 = (2\mu k^2 - d_1),\ \varepsilon_2 = (2\mu k^2 - d_2)$$
$$\varepsilon_3 = \beta_0^2 k^2/3\alpha,\ \lambda_5, d_3 = \rho_\omega\omega^2\delta/\lambda_3. \tag{36}$$

If we neglect the stretch effect, we obtain the period equation as

$$d\left[bd_1\lambda_2 - ad_2\lambda_1\frac{\tanh(\lambda_2 H)}{\tanh(\lambda_1 H)}\right] - 4\mu^2k^2(b-a)\delta$$
$$\times\lambda_1\lambda_2\frac{\tanh(\delta H)}{\tanh(\lambda_1 H)} + d_3\left[b\varepsilon_1\lambda_2 - a\varepsilon_2\lambda_1\frac{\tanh(\lambda_2 H)}{\tanh(\lambda_1 H)}\right]\tanh(\delta H) = 0 \tag{37}$$

which is the equation obtained by Sengupta and Benerji [10].

5. Antisymmetric vibrations

In this case, we consider that displacement u_1, stresses t_{11}, t_{33}, couple stress m_{32} and first moment β_3 are antisymmetric with respect to the plane $z=0$. Thus, we have from (14) to (17)

$$\Phi = A\sinh(\delta z)\exp[i(kx-\omega t)], \tag{38}$$

$$\Psi = [D\cosh(\lambda_1 z) + F\cosh(\lambda_2 z)]\exp[i(kx-\omega t)], \tag{39}$$

$$\omega_2 = [D_1\cosh(\lambda_1 z) + F_1\cosh(\lambda_2 z)]\exp[i(kx-\omega t)], \tag{40}$$

$$\Phi^* = M\sinh(\lambda_5 z)\exp[i(kx-\omega t)]. \tag{41}$$

Substituting the values of the potentials from (19), (38)–(41) in the boundary conditions (24), using the relation (4), (5), (22), (23) and taking $G = -p$, we obtain

$$d\sinh(\delta H)A + 2i\mu k\lambda_1 \sinh(\lambda_1 H)D + 2i\mu k\lambda_2 \sinh(\lambda_2 H)F - \rho_\omega \omega^2 \exp(-\lambda_3 H)G = 0, \tag{42}$$

$$2i\mu k\delta \cosh(\delta H)A - d_1 \cosh(\lambda_1 H)D - d_2 \cosh(\lambda_2 H)F = 0, \tag{43}$$

$$(\gamma+\varepsilon)\lambda_1 a \sinh(\lambda_1 H)D + (\gamma+\varepsilon)\lambda_2 b \sinh(\lambda_2 H)F - i\beta_0 k \sinh(\lambda_5 H)M = 0, \tag{44}$$

$$\delta \cosh(\delta H)A + ik\cosh(\lambda_1 H)D + ik\cosh(\lambda_2 H)F - \lambda_3 \exp(-\lambda_3 H)G = 0, \tag{45}$$

$$i\beta_0 ka\cosh(\lambda_1 H)D + i\beta_0 kb\cosh(\lambda_2 H)F 3\alpha_0 \lambda_5 \cosh(\lambda_5 H)M = 0. \tag{46}$$

Eliminating the constants, A, D, F, G and M from the above equations, we obtain the wave velocity equation for the antisymmetric vibrations of a micropolar elastic layer with stretch immersed in an infinite liquid as

$$\begin{aligned} &d(\gamma+\varepsilon)[bd_1\lambda_2 \tanh(\lambda_2 H) - ad_2\lambda_1 \tanh(\lambda_1 H)]\tanh(\delta H) \\ &- 4\mu^2 k^2(\gamma+\varepsilon)(b-a)\delta\lambda_1\lambda_2 \tanh(\lambda_1 H)\tanh(\lambda_2 H) \\ &+ (\gamma+\varepsilon)d_3[b\varepsilon_1\lambda_2 \tanh(\lambda_2 H) - a\varepsilon_2\lambda_1 \tanh(\lambda_1 H)] \\ &- d\varepsilon_3(bd_1 - ad_2)\tanh(\delta H)\tanh(\lambda_5 H) - 4\mu^2 k^2 \delta\varepsilon_3 \\ &\times [a\lambda_2 \tanh(\lambda_2 H) - b\lambda_1 \tanh(\lambda_1 H)]\tanh(\lambda_5 H) - \varepsilon_3 d_3 \\ &\times (b\varepsilon_1 - a\varepsilon_2)\tanh(\lambda_5 H). \end{aligned} \tag{47}$$

Eliminating the stretch effect, we obtain the equation as

$$d\left[bd_1\lambda_2 - ad_2\lambda_1 \frac{\tanh(\lambda_1 H)}{\tanh(\lambda_2 H)}\right]\tan(\delta H) - 4\mu^2 k^2(b-a)\delta\lambda_1\lambda_2 \tanh(\lambda_1 H) + d_3\left[b\varepsilon_1\lambda_2 - a\varepsilon_2\lambda_1 \frac{\tanh(\lambda_1 H)}{\tanh(\lambda_2 H)}\right] = 0. \tag{48}$$

This is in agreement with the result obtained by Sengupta and Benerji [10].

6. Discussions of the period equations

Case I. Here we assume that the wavelength is large compared to the thickness of the plate. In such a case, (35) and (47) reduce to

$$\begin{aligned} &d(\gamma+\varepsilon)(bd_1 - ad_2) - 4\mu^2 k^2(\gamma+\varepsilon)(b-a)\delta^2 \\ &+ (\gamma+\varepsilon)d_3\delta H(b\varepsilon_1 - a\varepsilon_2) - \frac{4\mu^2 k^2 \varepsilon_3 \delta^2 (a-b)}{\lambda_5} \\ &- \frac{d_3\varepsilon_3\delta}{\lambda_5}(b\varepsilon_1 - a\varepsilon_2) - \frac{d\varepsilon_3}{\lambda_5}(bd_1 - ad_2), \end{aligned} \tag{49}$$

$$(\gamma+\varepsilon)dr_1\left[\frac{bd_1}{s_1\lambda_1^2}-\frac{ad_2}{s_2\lambda_2^2}\right]-4\mu^2k^2(\gamma+\varepsilon)(b-a)$$
$$+\frac{(\gamma+\varepsilon)d_3}{\delta H}\left[\frac{b\varepsilon_1}{s_1\lambda_1^2}-\frac{a\varepsilon_2}{s_2\lambda_2^2}\right]-\frac{d\varepsilon_3 r_1 s_3\lambda_5}{s_1 s_2\lambda_1^2\lambda_2^2}(bd_1-ad_2)$$
$$-4\mu^2k^2\varepsilon_3 s_3\lambda_5\left[\frac{a}{s_1\lambda_1^2}-\frac{b}{s_2\lambda_2^2}\right]-\frac{\varepsilon_3 d_3 s_3\lambda_5}{\delta H s_1 s_2\lambda_1^2\lambda_2^2}(b\varepsilon_1-a\varepsilon_2), \tag{50}$$

where

$$r_1=\left(1-\frac{\delta^2H^2}{3}\right),\quad s_i=1-\frac{\lambda_1^2H^2}{3},\quad i=1,2,3. \tag{51}$$

Thus (49) and (50) represent the wave velocity equation for the longitudinal vibration of micropolar elastic layer with stretch immersed in an infinite liquid for symmetric and antisymmetric vibrations.

If we neglect the stretch effect, we obtain the corresponding results as

$$d(bd_1-ad_2)-4\mu^2k^2\delta^2(b-a)+\delta Hd_3(b\varepsilon_1-a\varepsilon_2)=0, \tag{52}$$

$$dr_1\left(\frac{bd_1}{s_1\lambda_1^2}-\frac{ad_2}{s_2\lambda_2^2}\right)-4\mu^2k^2(b-a)+\frac{d_3}{\delta H}\left(\frac{b\varepsilon_1}{s_1\lambda_1^2}-\frac{a\varepsilon_2}{s_2\lambda_2^2}\right)=0. \tag{53}$$

These are in agreement with the results obtained by Sengupta and Benerji [10].

Case II. Here we assume that the wavelength is small in comparison with the thickness of the plate. In this case (35) and (47) for symmetric and antisymmetric vibrations reduce to

$$d(\gamma+\varepsilon)(bd_1\lambda_2-ad_2\lambda_1)-4\mu^2k^2(\gamma+\varepsilon)(b-a)\delta\lambda_1\lambda_2$$
$$-d\varepsilon_3(bd_1-ad_2)-4\mu^2k^2\delta\varepsilon_3(a\lambda_2-b\lambda_1)+(\gamma+\varepsilon)d_3$$
$$\times(b\varepsilon_1\lambda_2-a\varepsilon_2\lambda_1)-\varepsilon_3d_3(b\varepsilon_1-a\varepsilon_2). \tag{54}$$

We note that the first part of (54) represents the expression for the modified Rayleigh waves in a micropolar medium with stretch and the second part represents the effect of the liquid on the propagation of the modified Rayleigh waves. Thus, the above expression corresponds to the modified Rayleigh waves on the floor (if we suppose the floor to be a micropolar medium with stretch) of the ocean.

By neglecting the stretch effect, we can easily obtain

$$d(bd_1\lambda_2-ad_2\lambda_1)-4\mu^2k^2(b-a)\delta\lambda_1\lambda_2+(b\varepsilon_1\lambda_2-a\varepsilon_2\lambda_1)d_3. \tag{55}$$

This agrees with the one obtained by Sengupta and Benerji [10].

7. Particular case

In this section, we discuss the propagation of monochromatic waves in an infinite micropolar elastic layer with stretch. We consider the monochromatic waves

propagating along the x-axis of a micropolar elastic layer with stretch of thickness $2H$. We assume that edges of the layer $z = \pm H$ are free of stresses. In this case the boundary conditions are

$$t_{33} = t_{31} = m_{32} = \beta_3 = 0 \quad \text{at} \quad z = \pm H, \tag{56}$$

where $t_{33}, t_{31}, m_{32}, \beta_3$ are given by (22).

Case I. Symmetric vibrations.

Substituting the values of Φ, Ψ, ω_2 and Φ^* from (25)–(28) in the boundary conditions (56) and making use of (22), we obtain the wave velocity equation as

$$\begin{aligned} & d(\gamma+\varepsilon)\left[bd_1\lambda_2 - ad_2\lambda_1 \frac{\tanh(\lambda_2 H)}{\tanh(\lambda_1 H)} \right] \\ & - 4\mu^2 k^2 (\gamma+\varepsilon)(b-a)\delta\lambda_1\lambda_2 \frac{\tanh(\delta H)}{\tanh(\lambda_1 H)} \\ & - d\varepsilon_3 (bd_1 - ad_2) \frac{\tanh(\lambda_2 H)}{\tanh(\lambda_5 H)} \\ & - 4\mu^2 k^2 \delta\varepsilon_3 \left[a\lambda_2 - b\lambda_1 \frac{\tanh(\lambda_2 H)}{\tanh(\lambda_1 H)} \right] \frac{\tanh(\delta H)}{\tanh(\lambda_5 H)}. \end{aligned} \tag{57}$$

By neglecting the stretch effect, we can obtain

$$d\left[bd_1\lambda_2 - ad_2\lambda_1 \frac{\tanh(\lambda_2 H)}{\tanh(\lambda_1 H)} \right] = 4\mu^2 k^2 (b-a)\delta\lambda_1\lambda_2 \frac{\tanh(\delta H)}{\tanh(\lambda_1 H)}. \tag{58}$$

This agrees with the result obtained by Nowacki [6].

Case II. Antisymmetric vibrations

Substituting for Φ, Ψ, ω_2 and Φ^* from (38)–(41) in (56) and (22), we obtain the wave velocity equation as

$$\begin{aligned} & d(\gamma+\varepsilon)[bd_1\lambda_2 \tanh(\lambda_2 H) - ad_2\lambda_1 \tanh(\lambda_1 H)] \tanh(\delta H) \\ & - 4\mu^2 k^2 (\gamma+\varepsilon)(b-a)\delta\lambda_1\lambda_2 \tanh(\lambda_1 H)\tanh(\lambda_2 H) \\ & - d\varepsilon_3 (bd_1 - ad_2)\tanh(\lambda_5 H)\tanh(\delta H) \\ & - 4\mu^2 k^2 \delta\varepsilon_3 [a\lambda_2 \tanh(\lambda_2 H) - b\lambda_1 \tanh(\lambda_1 H)] \tanh(\lambda_5 H). \end{aligned} \tag{59}$$

If we neglect the stretch effect, we can obtain the corresponding expression as

$$d\left(\frac{bd_1\lambda_2}{\tanh(\lambda_1 H)} - \frac{ad_2\lambda_1}{\tanh(\lambda_2 H)} \right) \tanh(\delta H) = 4\mu^2 k^2 (b-a)\delta\lambda_1\lambda_2. \tag{60}$$

This is in agreement with the result obtained by Nowacki [6].

8. Discussion of the period equations

Case I. When the wavelength is large compared to the thickness of the layer, then the quantities δH, $\lambda_1 H$, $\lambda_2 H$ and $\lambda_5 H$ are small. Therefore from (57) and (59), we can easily obtain

$$d(\gamma+\varepsilon)(bd_1-ad_2)-4\mu^2k^2(\gamma+\varepsilon)(b-a)\delta^2$$
$$-\frac{d\varepsilon_3}{\lambda_5}(bd_1-ad_2)-\frac{4\mu^2k^2\delta^2\varepsilon_3}{\lambda_5}(a-b)=0, \tag{61}$$

and

$$(\gamma+\varepsilon)dr_1\left[\frac{bd_1}{s_1\lambda_1^2}-\frac{ad_2}{s_2\lambda_2^2}\right]-4\mu^2k^2(\gamma+\varepsilon)(b-a)$$
$$-\frac{d\varepsilon_3 r_1 s_3\lambda_5}{s_1 s_2\lambda_1^2\lambda_2^2}(bd_1-ad_2)-4\mu^2k^2\varepsilon_3 s_3\lambda_3\left[\frac{a}{s_1\lambda_1^2}-\frac{b}{s_2\lambda_2^2}\right]. \tag{62}$$

Equations (61) and (62) represent the wave velocity equation for the longitudinal vibration of micropolar elastic layer with stretch.

By neglecting the stretch effect, we obtain

$$d(bd_1-ad_2)=4\mu^2k^2\delta^2(b-a), \tag{63}$$

$$dr_1\left(\frac{bd_1}{s_1\lambda_1^2}-\frac{ad_2}{s_2\lambda_2^2}\right)=4\mu^2k^2(b-a). \tag{64}$$

The results (63) and (64) agree with those obtained by Nowacki [6].

Case II. If the wavelength is small compared to the thickness of the layer, then we obtain from (57) and (59)

$$d(\gamma+\varepsilon)(bd_1\lambda_2-ad_2\lambda_1)-4\mu^2k^2(\gamma+\varepsilon)(b-a)\delta\lambda_1\lambda_2$$
$$-d\varepsilon_3(bd_1-ad_2)-4\mu^2k^2\delta\varepsilon_3(a\lambda_2-b\lambda_1)=0. \tag{65}$$

Equation (65) represents the expression for the modified Rayleigh waves in a micropolar elastic medium with stretch.

By neglecting the stretch effect, we obtain

$$d(bd_1\lambda_2-ad_2\lambda_1)=4\mu^2k^2(b-a)\delta\lambda_1\lambda_2. \tag{66}$$

The result (66) agrees with the result obtained by Nowacki [6].

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A descriptive version of Ambrose's representation theorem for flows

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Abstract. We prove here an analogue of Ambrose–Kakutani representation theorem for measurable flows. No measure is required and no points are dropped. This helps us to generalize a theorem due to Shelah and Weiss and answer a question due to A Ramsay.

Keywords. Measurable flow; flow built under a function; ergodic measures; Polish topology.

1. Introduction

Recently there has been an attempt to see how much of ergodic theory can be done in the setting of a general measurable structure. In [6], Weiss confined himself to the case of actions of $\mathbb{Z}$, i.e. the iterates of a single Borel automorphism of a standard Borel space. It is natural to see how many of the known things admit a generalization to a measurable setting in the case of actions of other groups.

In the case of action of $\mathbb{R}$, Ambrose [1] proved the existence of a prototype for ergodic flows preserving a measure, the so-called flow built under a function. This result has been extended to the case of non-singular flows by Dani [2] and Krengel [3]. However, in all these cases, the result is proved modulo the null sets of a measure on the space under consideration. We prove here that the flow built under a function is a prototype for all measurable flows satisfying a certain condition (see theorem 1). It turns out that every jointly measurable flow (on a standard Borel space) which does not admit a fixed point satisfies the condition of our theorem.

Some interesting consequences of our theorem are (i) every jointly measurable flow on a standard Borel space is a flow of homeomorphisms on a complete separable metric space (see theorem 2 and its corollary). This answers a question of Ramsay [5] for flows indexed by the real line. (ii) Every jointly measurable flow without fixed points admits, for each α, $0 \leqslant \alpha \leqslant 1$, a Borel set A_α such that every orbit spends precisely α proportion of time in A_α. (iii) Every jointly measurable flow (on a standard Borel space) whose orbit space does not admit a Borel cross-section is strictly ergodic (i.e. non-transitive and ergodic) with respect to some continuous finite measure on the Borel space. This extends to the case of $\mathbb{R}$ actions a result of Shelah and Weiss for Z-actions.

Since this paper was submitted for publication, [4] has been written which gives a connected account of ergodic theory in general measurable structures.

2.

Let $(X, \mathscr{B})$ be a standard Borel space. Let $T_t, t\in\mathbb{R}$ be a jointly measurable flow of Borel automorphisms on X, i.e. for each $t\in\mathbb{R}$, T_t is a Borel automorphism of X such that

(i) the map $(t,x)\to T_t x$ from $\mathbb{R}\times X\to X$ is measurable,
(ii) $T_0 x = x$ for all x,
(iii) $T_{t+s}x = T_t(T_s x)$.

(Here $\mathbb{R}$ is given the usual Borel structure and $\mathbb{R}\times X$ the product Borel structure.) The purpose of this paper is to prove the following theorem and give some of its applications.

Theorem 1. *Let $(X, \mathscr{B})$, $T_t, t\in\mathbb{R}$, be as above. Assume that there is a set $B\in\mathscr{B}$ such that for all $x\in X$ the sets $\{t\in\mathbb{R}: T_t x\in B\}$ and $\{t\in\mathbb{R}: T_t x\notin B\}$ have positive Lebesgue measure. Then there is a Borel set $\Omega_0\subseteq X$ such that for every $x\in X$ the set $\{t: T_t x\in\Omega_0\}$ is non-empty and discrete, hence either finite or countably infinite.*

Proof. The method of proof is adapted from Ambrose's original paper where a measure preserving flow on a finite measure space is shown to be isomorphic to a flow built under a function (see [1]).

(i) Let $A = X - B$ and let U denote the characteristic function of A. Write

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\int_0^\varepsilon U(T_t x)\,dt.$$

Now for any $s\in\mathbb{R}$

$$\begin{aligned}\varphi_\varepsilon(T_s x) &= \frac{1}{\varepsilon}\int_0^\varepsilon U(T_t T_s x)\,dt\\ &= \frac{1}{\varepsilon}\int_0^\varepsilon U(T_{t+s}x)\,dt\\ &= \frac{1}{\varepsilon}\int_s^{s+\varepsilon} U(T_t x)\,dt \to U(T_s x),\end{aligned}$$

as $\varepsilon\to 0$ for almost every s with respect to Lebesgue measure. This is because of Lebesgue's density theorem. By hypothesis, therefore, for every x, there exists s_1, s_2 such that $\varphi_\varepsilon(T_{s_1}x)\to 0$ and $\varphi_\varepsilon(T_{s_2}x)\to 1$ as $\varepsilon\to 0$.

(ii) The function $\xi(s) = \varphi_\varepsilon(T_s x)$, for fixed ε and x is Lipschitz with constant $2/\varepsilon$. Indeed

$$\begin{aligned}\xi(s) - \xi(l) &= \frac{1}{\varepsilon}\int_0^\varepsilon U(T_{t+s}x)\,dt - \frac{1}{\varepsilon}\int_0^\varepsilon U(T_{t+l}x)\,dt\\ &= \frac{1}{\varepsilon}\left(\int_s^{s+\varepsilon} U(T_t x)\,dt - \int_l^{l+\varepsilon} U(T_t x)\,dt\right)\\ &= \frac{1}{\varepsilon}\left(\int_s^{l} U(T_t x)\,dt - \int_{s+\varepsilon}^{l+\varepsilon} U(T_t x)\,dt\right).\end{aligned}$$

Since U is bounded by one we see that

$$|\xi(s) - \xi(l)| \leqslant \frac{2}{\varepsilon}|l - s|.$$

(iii) Let

$$A_n = \left\{x : \frac{1}{1/n}\int_0^{1/n} U(T_s x) < \tfrac{1}{4}\right\},$$

$$B_n = \left\{x : \frac{1}{1/n}\int_0^{1/n} U(T_s x) > \tfrac{3}{4}\right\}.$$

Let A_n^*, B_n^* be the smallest sets invariant under the flow and containing A_n and B_n respectively. Let $Q_n = A_n^* \cap B_n^*$. The sets A_n and B_n are Borel and from the manner in which they are defined we see that for any x the sets $\{t : T_t x \in A_n\}$, $\{t : T_t x \in B_n\}$ are open in $\mathbb{R}$. We see from this that $A_n^* = \cup T_r A_n$, $B_n^* = \cup T_r B_n$ where the unions are taken over r running over rational numbers. Thus A_n^*, B_n^* are also Borel sets. The sets A_n^*, B_n^* are invariant under the flow, hence Q_n is invariant under the flow . In view of the conclusion under 1, each $x \in X$ belongs to some Q_n. Write

$$\begin{aligned} P_1 &= Q_1 \\ P_2 &= Q_2 - Q_1 \\ &\vdots \quad \vdots \\ P_n &= Q_n - (Q_1 \cup Q_2 \cup \cdots \cup Q_{n-1}) \\ &\vdots \quad \vdots \quad \vdots \end{aligned}$$

The sets $P_1, P_2, P_3, \ldots$ are pairwise disjoint, measurable, invariant under the flow, and their union is all of X.

(iv) Let

$$\Omega_n^1 = \left\{x : \varphi_{1/n}(x) = \tfrac{1}{2}, \varphi_{1/n}(T_s x) < \tfrac{1}{2} \text{ for } 0 < s < \frac{1}{8n}\right\}$$

$$\Omega_n^2 = \left\{x : \varphi_{1/n}(x) = \tfrac{1}{2}, \varphi_{1/n}(T_s x) > \tfrac{1}{2} \text{ for } 0 < s < \frac{1}{8n}\right\}$$

$$\Omega_n = \Omega_n^1 \cup \Omega_n^2.$$

In view of the definition of Ω_n, it is clear that for any x the set $\{t : T_t x \in \Omega_n\} = E_n(x)$ is discrete; two points of $E_n(x)$ being separated by a distance at least $1/8n$. We show that the orbit of every $x \in P_n$ intersects Ω_n. Let $x \in P_n$. Then there exists s_1 and s_2 such that

$$\varphi_{1/n}(T_{s_1} x) < \tfrac{1}{4} \quad \text{and} \quad \varphi_{1/n}(T_{s_2} x) > \tfrac{3}{4}.$$

The function $\varphi_{1/n}(T_s x)$ as a function of s is Lipschitz, hence continuous. Hence there exists a largest s_0 between s_1 and s_2 such that $\varphi_{1/n}(T_{s_0} x) = \frac{1}{2}$. We show that $T_{s_0} x \in \Omega_n$. Now s_0 is strictly between s_1 and s_2. If $s_0 < s_1$ we see that

(i) $\varphi_{1/n}(T_s x) < \frac{1}{2}$ for $s_0 < s < s_1$ i.e.

$$\varphi_{1/n}(T_s T_{s_0} x) < \tfrac{1}{2} \quad \text{for } 0 < s < s_1 - s_0.$$

(ii) Since $\varphi_{1/n}(T_{s_0}x) = \frac{1}{2}$ and $\varphi_{1/n}(T_{s_1}x) < \frac{1}{4}$, we see that

$$\tfrac{1}{4} < |\varphi_{1/n}(T_{s_0}x) - \varphi_{1/n}(T_{s_1}x)| < 2n|s_0 - s_1|$$

i.e. $1/8n < s_1 - s_0$.

Thus (i) and (ii) show that $T_{s_0}x \in \Omega_n^1$, hence in Ω_n. If $s_0 < s_2$, then we see similarly that $T_{s_0}x \in \Omega_n^2$, hence in Ω_n. Thus orbit of x intersects Ω_n. Obviously the set $E_n(x) = \{t: T_t x \in \Omega_n\}$ is discrete. If we set

$$\Omega_0 = \bigcup_{n=1}^{\infty} (P_n \cap \Omega_n)$$

then Ω_0 is measurable and for any $x \in X$ the set $\{t: T_t x \in \Omega_0\}$ is non-empty and discrete. This proves the theorem.

3.

(i) We continue with the assumptions of theorem 1. Let $\Omega_0 \subseteq X$ be a Borel set such that for each $x \in X$ the set $E(x) = \{t: T_t x \in \Omega_0\}$ is non-empty and discrete. The subset of $\mathbb{R} \times X$ defined by $\{(t, x): T_t x \in \Omega_0\}$ is a Borel set such that each x section, is non-empty and discrete in $\mathbb{R}$. The functions

$$i(x) = \inf\{t: T_t x \in \Omega_0\},$$
$$j(x) = \sup\{t: T_t x \in \Omega_0\},$$

are then measurable functions. The sets

$$I = \{x: i(x) > -\infty\}$$
$$J = \{x: j(x) < \infty\}$$

are measurable and invariant under the flow. Further

$$i(T_t x) = i(x) - t \tag{1}$$

and

$$j(T_t x) = j(x) - t. \tag{2}$$

If we write $k(x) = T_{i(x)}x$ for $x \in I$ then $k(T_t x) = k(x)$ because of (1). The function $k: I \to I$ is thus measurable and constant on orbits. This means that if we restrict the flow to I then its orbit space admits a Borel cross-section. Thus on the set $I \cup J$ the orbit space of the flow admits a Borel cross-section. It is easy to see that on the set $K = I \cup J$, the flow is isomorphic to a flow built under a function, indeed, under a constant function.

(ii) Now let $\Omega = (X - K) \cap \Omega_0$. Define for $w \in \Omega$,

$$Sw = T_{s(w)}w,$$

where $s(w)$ is the smallest $t > 0$ such that $T^{s(w)}w \in \Omega$. Since $w \notin K, i(w) = -\infty$ and $j(w) = \infty$, so that S is one-one and onto and Borel as a function from Ω to Ω. We

define $f(w) = s(w)$. Then it is clear that the flow T_t, $t \in \mathbb{R}$ is isomorphic to the flow built under the function f with base space Ω and base transformation S. Thus we have the following corollary of theorem 1.

COROLLARY

Every flow $T_t, t \in \mathbb{R}$ on a standard Borel space satisfying the assumptions of theorem 1 is isomorphic to a flow built under a function.

Remark. A result of Shelah and Weiss [6] states that if the orbit space of a Borel automorphism S on a standard Borel space does not admit a Borel cross-section then the space admits a continuous probability measure which is quasi-invariant and ergodic under S. It is easy to see that the corollary above permits us to extend this result to the case of jointly measurable flows on a standard Borel space.

4.

Let $(\Omega, \mathscr{B}_0)$ be a standard Borel space, $S:\Omega \to \Omega$ a Borel automorphism, and, $f:\Omega \to \mathbb{R}$ a positive measurable function such that for $w \in \Omega$ the sums $\sum_{k=0}^{\infty} f(S^k w)$ and $\sum_{k=0}^{\infty} f(S^{-k} w)$ are infinite. One can then define a flow $T_t, t \in \mathbb{R}$ on the set

$$X = \{(w, t): 0 \leqslant t < f(w)\}$$

the so-called flow built under the function f with base transformation S and base space Ω. The set X is given the Borel structure it inherits as a subset of $\Omega \times \mathbb{R}$.

A theorem proved in Weiss [6] and attributed by him to Ramsay and Mackey may be stated as follows:

Theorem. *Given a Borel automorphism S_1 on Ω and a countable collection τ of sets in $\mathscr{B}_0$, there is a Polish topology on Ω such that*

(i) *S_1 is a homeomorphism under this topology*
(ii) *every member of τ is open in this topology*
(iii) *the Borel sets generated by this topology from precisely the class $\mathscr{B}_0$.*

One can use this theorem to give a Polish topology $\mathscr{T}$ on Ω such that S is a homeomorphism under $\mathscr{T}$, f is continuous, and, Borel sets of $\mathscr{T}$ agree with the class $\mathscr{B}_0$. Let Ω be given such a topology $\mathscr{T}$. We next equip $\Omega \times \mathbb{R}$ with product topology, (where $\mathbb{R}$ is given the usual topology). Let $\bar{X} = \{(w, t): 0 \leqslant t \leqslant f(w)\}$ be the closure of X in $\Omega \times \mathbb{R}$. Define $g:\bar{X} \to X$ by

$$\begin{aligned} g(w, t) &= (w, t) \quad \text{if } 0 \leqslant t < f(w) \\ &= (Sw, 0) \quad \text{if } t = f(w) \end{aligned}$$

and let $\mathscr{T}_1$ be the largest topology on X which makes g continuous. It is clear that the flow $T_t, t \in \mathbb{R}$ (built under f with base transformation S and base space Ω) is a jointly continuous flow of homeomorphisms under this topology.

PROPOSITION

$\mathscr{T}_1$ is Polish.

Proof. (Suggested by M S Raghunathan). There is no loss of generality if we assume that $f(w) = 1$ for all $w \in \Omega$. For otherwise the transformation

$$J(w, s) = (w, s/f(w))$$

from X onto $\Omega x[0, 1)$ will reduce the problem to the case where $f(w) = 1$ for all w.

Let Y denote the countable cartesian product of $\mathbb{R}$ with itself with product topology. Let $f: \Omega \to Y$ be a homeomorphic imbedding of Ω in Y, as is possible by Urysohn's metrization theorem. Let W denote the cartesian product $\prod_{i \in Z} Y_i$ where for each integer i, $Y_i = Y$.

Give W the product topology and imbed Ω into W by the map

$$\xi(x) = (f(S^n x))_{n \in Z}.$$

Then $\xi(Sx) = S_1(\xi(x))$ where S_1 denotes the backward shift on W, i.e. the nth co-ordinate of $S_1 w$ is the $(n+1)$th co-ordinate of w. Now Y and W are real linear spaces. Further they are closed sets in Y_c and W_c, where Y_c and W_c are formed from the complex numbers $\mathbb{C}$ the same way the spaces Y and W were formed from $\mathbb{R}$. For $0 \leqslant t < 1$ define the map $w \to w^t$ from $W \to W$ as follows:

If $w = (w_n)_{n \in Z}$, then the nth co-ordinate of w^t is given by $(w^t)_n = (1-t)w_n + tw_{n+1}$. Clearly $w^0 = w$ and $w^t \to S_1 w$ as $t \to 1$. Now consider the map $h: W \times [0, 1) \to W_c \times \mathbb{C}$ given by $h(w, t) = (w_t + (1 - \exp(2\pi i t))iw, 1 - \exp(2\pi i t))$. For each t, $h(., t)$ is a homeomorphism of W into $W_c \times \mathbb{C}$. Further h is one-one, continuous and satisfies

(i) $h(w, 0) = (w, 0)$ for all $w \in W$,

(ii) $\lim_{t \to 1} h(w, t) = (S_1 w, 0)$ for all $w \in W$.

The restrictions h to $W \times [0, a]$ and $W \times [a, 1)$ are even homeomorphisms provided $0 < a < 1$. The map $k(x, t) = h(\xi(x), t)$, $(x, t) \in \Omega x[0, 1)$ is a homeomorphism of $\Omega \times [0, 1)$ onto its image under k provided $\Omega \times [0, 1)$ is equipped with topology $\mathscr{T}_1$. Now if we show that the image of $\Omega \times [0, 1)$ under the map k is G_δ in $W_c \times \mathbb{C}$, then it will follow that the topology $\mathscr{T}_1$ is Polish. However

$$k(\Omega \times [0, 1)) = k(\Omega \times [0, \tfrac{1}{2}]) \cup k(\Omega \times [\tfrac{1}{2}, 1)).$$

Each term on the right hand side is G_δ because k is a homeomorphism on $\Omega \times [0, \frac{1}{2}]$ and $\Omega \times [\frac{1}{2}, 1)$, where these sets are equipped with product topology. Thus the image under k of $\Omega \times [0, 1)$ is G_δ. q.e.d.

We have thus proved:

Theorem 2. *If T_t, $t \in \mathbb{R}$, is a jointly measurable flow on a standard Borel space $(X, \mathscr{B})$ and if this flow is isomorphic to a flow built under a function then X admits a Polish topology under which $T_t, t \in \mathbb{R}$ is a jointly continuous flow of homeomorphisms and the Borel sets of the topology form precisely the class $\mathscr{B}$.*

COROLLARY

Let $T_t, t\in\mathbb{R}$, be a jointly measurable flow on a standard Borel space $(X,\mathscr{B})$, then there is a Polish topology $\mathscr{T}$ on X such that

(i) *the flow $T_t, t\in\mathbb{R}$ is jointly continuous on $\mathbb{R}\times X$.*

(ii) *the Borel sets generated by $\mathscr{T}$ form precisely the class $\mathscr{B}$.*

Proof. Let F denote the fixed points of the flow and $Y = X - F$. The flow restricted to Y has no fixed points and is thus isomorphic to a flow built under a function by corollary of theorem 1 mentioned in § 2. We can apply theorem 2 to the restriction of the flow to Y. Further the flow when restricted to F is jointly continuous with respect to any topology on F. The corollary therefore follows.

This corollary answers a question raised by Ramsay [5] at least for flows indexed by the real line.

Remark 1. Every jointly measurable flow $T_t, t\in\mathbb{R}$ which does not admit a fixed point satisfies the condition of the theorem of § 1. To see this let $A_1, A_2, A_3, \ldots$ be a set of generators of $\mathscr{B}$. We first observe that for each $x\in X$, there is an A_i such that the sets $\{t: T_t x\in A_i\}$ and $\{t: T_t x\notin A_i\}$ both have positive Lebesgue measure. For otherwise, for every A_i, one of the sets $\{t: T_t x\in A_i\}$ and $\{t: T_t x\notin A_i\}$ has zero Lebesgue measure and the other has full Lebesgue measure. We write

$$C_i = \begin{cases} A_i^c & \text{if } \{t: T_t x\in A_i \text{ has zero Lebesgue measure}\} \\ A_i & \text{otherwise.} \end{cases}$$

Then for almost every t, $T_t x\in \bigcap_{i=1}^{\infty} C_i$. But the set $\bigcap_{i=1}^{\infty} C_i$ is either empty or a singleton set by virtue of the collection $A_1, A_2, A_3 \ldots$ being generators for $\mathscr{B}$. Thus x is a fixed point contrary to assumption. Now let

$$E_i = \{x: T_t x\in A_i \text{ for a } t \text{ set of positive Lebesgue measure.}\}$$

$$F_i = \{x: T_t x\notin A_i \text{ for a } t \text{ set of positive Lebesgue measure.}\}$$

The sets E_i, F_i are invariant sets under the flow $T_t, t\in\mathbb{R}$ and they are measurable. We let $G_i = F_i\cap E_i$ and set $H_1 = G_1$, $H_2 = G_2 - G_1, \ldots, H_n = G_n - \bigcup_{k=1}^{n-1} G_k \ldots$. The sets H_1 are invariant, disjoint and their union is all of X. Let $B = (A_1\cap H_1)\cup(A_2\cap H_2)\cup\cdots\cup(A_n\cap H_n)\cup\cdots$ Then B is in $\mathscr{B}$ and for every $x\in X$, the sets $\{t: T_t x\in B\}$ and $\{t: T_t x\notin B\}$ have Lebesgue measure positive. The flow $T_t, t\in\mathbb{R}$, without fixed points, thus satisfied the condition of theorem 1.

Remark 2. Using the isomorphism with a flow built under a function we can show that given a jointly measurable flow $T_t, t\in\mathbb{R}$ without fixed points and a real number α, $0\leqslant\alpha\leqslant 1$, there exists a measurable set B such that for every x the orbit of x spends α proportion of time in B, more precisely, $1/N$ Lebesgue measure

$$\{t: T_t x\in B, 0\leqslant t\leqslant N\}\to\alpha \quad \text{as } N\to\infty.$$

Remarks 1 and 2 were suggested by M G Nadkarni.

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Generalized epimorphism theorem

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Abstract. Let $R[X, Y]$ be a polynomial ring in two variables over a commutative ring R and let $F \in R[X, Y]$ such that $R[X, Y]/(F) = R[Z]$ (a polynomial ring in one variable). In this set-up we prove that $R[X, Y] = R[F, G]$ for some $G \in R[X, Y]$ if either R contains a field of characteristic zero or R is a seminormal domain of characteristic zero.

Keywords. Epimorphism theorem; polynomial ring; seminormal domain; characteristic zero.

1. Introduction

Let k be a field of characteristic zero. Let $k[X, Y]$ be a polynomial ring in two variables over k and $F \in k[X, Y]$ such that $k[X, Y]/(F) = k[Z]$ (a polynomial ring in one variable). In this set-up the famous epimorphism theorem of Abhyankar and Moh ([2], Theorem 1.2) says that $k[X, Y] = k[F, G]$ for some $G \in k[X, Y]$. Russell and Sathaye had obtained the following analog of the epimorphism theorem ([6], Theorem 2.6.2): If R is a locally factorial Krull domain of characteristic zero and $F \in R[X, Y]$ such that $R[X, Y]/(F) = R[Z]$, then $R[X, Y] = R[F, G]$. Therefore one asks the following natural question:

Is the foregoing result valid for an arbitrary commutative domain R of characteristic zero?

In this paper we answer this question affirmatively under the assumption that R is seminormal. We prove:

Theorem A. *Let R be a seminormal commutative domain of characteristic zero. Let I be an ideal of $R[X, Y]$ such that $R[X, Y]/I = R[Z]$. Then I is a principal ideal say generated by F and $R[X, Y] = R[F, G]$ for some $G \in R[X, Y]$.*

Moreover we give an example (Example 3.8) to show that I need not be principal if R is not seminormal. When R contains a field of characteristic zero we prove the following (weaker) epimorphism theorem:

Theorem B. *Let R be a commutative ring containing a field of characteristic zero. Let $F \in R[X, Y]$ such that $R[X, Y]/(F) = R[Z]$. Then $R[X, Y] = R[F, G]$ for some $G \in R[X, Y]$.*

2. Preliminaries

Throughout this paper all rings will be commutative.

In this section we set up notations and state some results for later use.

R will denote a commutative ring.

$R^{[n]}$: polynomial ring in n variables over R.

R^n: free R-module of rank n.

For a finitely generated R-algebra A,

$\Omega_{A/R}$: universal module of R-differentials of A.

For a prime ideal $\mathfrak{P}$ of R,

$k(\mathfrak{P})$: $R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$.

DEFINITION

A reduced ring R is said to be *seminormal* if it satisfies the condition: for $b, c \in R$ with $b^3 = c^2$, there is an $a \in R$ with $a^2 = b$, $a^3 = c$.

Lemma (2.1). *Let R be a noetherian ring and let $s \in R$ be a non-zero divisor. Let M be a finitely generated R-module. If M_s is a projective R_s-module of rank d and M/sM is R/sR-projective of rank d then M is R-projective of rank d.*

Proof. Without loss of generality we can assume that R is local.

Since M/sM is R/s-projective and R is local there exists a surjective R-linear map β: $R^d \twoheadrightarrow M$ ($d = \operatorname{rank} M/sM$). Let $N = \ker \beta$. Since M_s is R_s-projective of rank d and β is surjective we get $N_s = 0$. But s is a non-zero-divisor of R and $N \subset R^d$, therefore $N_s = 0 \Rightarrow N = 0$ and β is an isomorphism.

Lemma (2.2). *Let R be a noetherian ring and I be an ideal of $R^{[n]}$ such that $R^{[n]}/I \approx R^{[n-1]}$ as R-algebras. Then for an ideal $\mathfrak{G}$ of R, $I \cap \mathfrak{G}R^{[n]} = \mathfrak{G}I$. Moreover if I is a principal ideal of $R^{[n]}$ say generated by F, then*

(i) *F is a non-zero-divisor of $R^{[n]}$.*

(ii) *F is a algebraically independent over R, i.e. $R[F] \approx R^{[1]}$.*

(iii) *$R[F] \cap \mathfrak{G}R^{[n]} = \mathfrak{G}R[F]$ for any ideal $\mathfrak{G}$ of R.*

Proof. Since for any non-negative integer l, $R^{[l]}$ is a free R-module, the exact sequence $0 \to I \to R^{[n]} \xrightarrow{\alpha} R^{[n-1]} \to 0$; α: R-algebra homomorphism of R-modules gives rise to the exact sequence

$$0 \to I \underset{R}{\otimes} R/\mathfrak{G} \to R^{[n]} \underset{R}{\otimes} R/\mathfrak{G} \xrightarrow{\alpha \otimes_{R^1} R/\mathfrak{G}} R^{[n-1]} \underset{R}{\otimes} R/\mathfrak{G} \to 0$$

proving that the canonical map $I/\mathfrak{G}I \to I + \mathfrak{G}R^{[n]}/\mathfrak{G}R^{[n]}$ of $R/\mathfrak{G}^{[n]}$-modules is an isomorphism. Hence $I \cap \mathfrak{G}R^{[n]} = \mathfrak{G}I$.

Now we assume that $I = (F)$.

(i) It is easy to see that $F \notin mR^{[n]}$ for any maximal ideal m of R. This shows that F is a non-zero-divisor of $R^{[n]}$.

(ii) Suppose $c_0 + c_1F + \cdots + c_rF^r = 0$ where $c_i \in R \forall i$, $0 \leqslant i \leqslant r$. Then $0 = \alpha(c_0 + c_1F + \cdots + c_rF^r) = c_0$ i.e. $F(c_1 + c_2F + \cdots + c_rF^{r-1}) = 0$. Therefore, as by (i) F is a non-zero-

divisor, $c_1 + c_2F + \cdots c_rF^{r-1} = 0$ showing that $c_1 = 0$. Repeating this argument we see that $c_i = 0 \forall i$, $0 \leqslant i \leqslant r$.
(iii) Let $\bar{F}$ be the image of F in $R/\mathfrak{G}^{[n]}$ $(= R^{[n]}/\mathfrak{G}R^{[n]})$. Then obviously $R/\mathfrak{G}^{[n]}/(\bar{F}) \approx R/\mathfrak{G}^{[n-1]}$. Therefore by (ii) $\bar{F}$ is algebraically independent over $R/\mathfrak{G}$ and hence $R[F] \cap \mathfrak{G}R^{[n]} = \mathfrak{G}R[F]$.

Lemma (2.3). *Let R be a noetherian ring and let $\mathfrak{G}$ be the nilradical of R. Let I be an ideal of $R^{[n]}$ such that $R^{[n]}/I \approx R^{[n-1]}$ as R-algebras. If $I/\mathfrak{G}I$ is a projective $R/\mathfrak{G}^{[n]}$-module of (constant) rank 1 then I is a projective $R^{[n]}$-module of (constant) rank 1.*

Proof. Since $\mathfrak{G}$ is nilpotent, the canonical map $\mathrm{Pic}(R^{[n]}) \to \mathrm{Pic}(R/\mathfrak{G}^{[n]})$ is an isomorphism. Therefore there exists a projective $R^{[n]}$-module L of constant rank 1 such that $L/\mathfrak{G}L \approx I/\mathfrak{G}I$. Hence there exists a $R^{[n]}$-linear map $\psi\colon L \to I$ such that the induced map $\bar{\psi}\colon L/\mathfrak{G}L \to I/\mathfrak{G}I$ is an isomorphism.
We claim that ψ is an isomorphism.

Surjectivity of ψ: Since $\bar{\psi}$ is an isomorphism, we have $I = \psi(L) + \mathfrak{G}I$. But $\mathfrak{G}$ is nilpotent and hence $I = \psi(L)$.

Injectivity of ψ: Let $M =$ ker ψ. Then we get the following exact sequence of $R^{[n]}$-modules:

$$0 \to M \to L \xrightarrow{\psi} I \to 0.$$

As in Lemma 2.2 we see that I is a projective R-module. Therefore the above exact sequence gives rise to the following exact sequence:

$$0 \to M/\mathfrak{G}M \to L/\mathfrak{G}L \xrightarrow{\bar{\psi}} I/\mathfrak{G}I \to 0.$$

But $\bar{\psi}$ is an isomorphism. Therefore $M/\mathfrak{G}M = 0$ i.e. $M = \mathfrak{G}M$. The nilpotency of $\mathfrak{G}$ shows that $M = 0$.
Thus ψ is an isomorphism.

3. Main theorems

In this section we prove Theorem A and Theorem B which are quoted in the introduction. For the proof of these theorems we need some lemmas and a proposition. Lemma 3.1 is well known but for the lack of a proper reference we give a proof.

Lemma 3.1. Let R be a noetherian ring and S be a noetherian R-algebra. Let $\pi \in R$ be such that S_π is a flat R_π-algebra and $S/\pi S$ is a flat R/π R-algebra. Moreover assume that $\mathrm{Tor}_1^R(S, R/\pi R) = 0$. Then S is a flat R-algebra.

Proof. Let M and N be finitely generated R-modules and let $f\colon M \to N$ be a R-linear injective map. Then we want to show that the map $f \otimes 1_S\colon M \otimes_R S \to N \otimes_R S$ is injective. Let $K = \ker(f \otimes 1_S)$.

Since S_π is R_π-flat we have $K_\pi = 0$. Let $T = 1 + \pi R$ and $T' = 1 + \pi S$. Then since $\mathrm{Tor}_1^{R_T}(S_{T'}, R_T/\pi R_T) = \mathrm{Tor}_1^R(S, R/\pi R) \otimes_S S_{T'} = 0$ and $R_T/\pi R_T = R/\pi R$, $S_{T'}/\pi S_{T'} = S/\pi S$, by ([1], Theorem 3.2, p. 91) $S_{T'}$ is flat over R_T and hence $K_{T'} = 0$.

Thus $K_{T'} = 0$, $K_\pi = 0$. Therefore $K = 0$ showing that S is a flat R-algebra.

Lemma 3.2. Let R be a noetherian ring of finite Krull dimension (denoted by dim R*). Let F be an element of $R[X, Y]$ such that $R[X, Y]/(F) = R[Z]$ as R-algebras. Then $R[X, Y]$ is a flat $R[F]$-algebra.*

Proof. Let $\mathfrak{G}$ be the nilradical of R. Since $R[F]$ (F being algebraically independent over R) and $R[X, Y]$ are flat over R, for every module M over R we have $\mathrm{Tor}_1^{R[F]}(R[X, Y], M \otimes_R R[F]) = 0$. In particular for every ideal J of R we have $\mathrm{Tor}_1^{R[F]}(R[X, Y], R[F]/JR[F]) = 0$. Therefore, since $\mathfrak{G}$ is nilpotent, by ([1], Theorem 3.2, p. 91) $R[X, Y]$ is flat over $R[F]$ if $R[X, Y]/\mathfrak{G}R[X, Y]$ ($= R/\mathfrak{G}[X, Y]$) is flat over $R[F]/\mathfrak{G}R[F]$. So it is enough to prove the result when R is a reduced ring.

We prove the result by induction on dim R. Without loss of generality we can assume that R is local.

If dim $R = 0$ then R is a field, $R[F]$ is a principal ideal domain and $R[X, Y]$ is a domain. Therefore $R[X, Y]$ is $R[F]$-flat.

Now we assume that dim $R > 0$. Let $\pi \in R$ be a nonunit non-zero-divisor of R. Let $\bar{F}$ denote the image of F in $R/(\pi)[X, Y]$. Then $R/(\pi)[\bar{F}] = R[F]/\pi R[F]$. Since dim $R/(\pi) <$ dim R and dim $R_\pi <$ dim R, by the induction hypothesis $R_\pi[X, Y]$ is flat over $R_\pi[F]$ and $R[X, Y]/\pi R[X, Y]$ is flat over $R[F]/\pi R[F]$. Moreover $\mathrm{Tor}_1^{R[F]}(R[X, Y], R[F]/\pi R[F]) = 0$. Therefore by Lemma 3.1 $R[X, Y]$ is a flat $R[F]$-algebra.

Thus the proof of Lemma 3.2 is complete.

We state a definition before stating the next lemma.

DEFINITION

An element F of $R[X, Y]$ is called a *residual variable* if for every prime ideal $\mathfrak{P}$ of R, $k(\mathfrak{P})[X, Y] = k(\mathfrak{P})[\bar{F}]^{[1]}$ where $\bar{F}$ denotes the image of F in $k(\mathfrak{P})[X, Y]$.

Lemma 3.3. Let R be a ring and $F \in R[X, Y]$ be such that $R[X, Y]/(F) = R[Z]$ as R-algebras. Assume that F is a residual variable. Then for every prime ideal $\mathfrak{P}'$ of $R[F]$, $k(\mathfrak{P}') \otimes_{R[F]} R[X, Y] = k(\mathfrak{P}')^{[1]}$.

Proof. Let $\mathfrak{P}' \cap R = \mathfrak{P}$. Then $\mathfrak{P}R[F] \subset \mathfrak{P}'$ and by (2.2) $\mathfrak{P}\,R[F] = \mathfrak{P}R[X, Y] \cap R[F]$. Since F is a residual variable, we have $k(\mathfrak{P})[X, Y] = k(\mathfrak{P})[\bar{F}]^{[1]}$ where $\bar{F}$ denotes the image of F in $k(\mathfrak{P})[X, Y]$. Moreover $k(\mathfrak{P})[\bar{F}] = k(\mathfrak{P}) \otimes_R R[F]$ and there exists a $R[F]$-algebra homomorphism $k(\mathfrak{P})[\bar{F}] \to k(\mathfrak{P}')$. Therefore

$$k(\mathfrak{P}') \underset{R[F]}{\otimes} R[X, Y] = k(\mathfrak{P}') \underset{k(\mathfrak{P})[\bar{F}]}{\otimes} k(\mathfrak{P})[\bar{F}] \underset{R[F]}{\otimes} R[X, Y]$$

$$= k(\mathfrak{P}')^{[1]}.$$

Lemma 3.4. Let R be a ring and $F \in R[X, Y]$ be such that $R[X, Y]/(F) = R[Z]$ as R-algebras. Assume that F is a residual variable. Then $\Omega_{R[X,Y]/R[F]}$ is a free $R[X, Y]$-module of rank one.

Proof. We have the following right exact sequence of $R[X, Y]$-modules

$$\Omega_{R[F]/R} \underset{R[F]}{\otimes} R[X, Y] \xrightarrow{\theta} \Omega_{R[X,Y]/R} \to \Omega_{R[X,Y]/R[F]} \to 0.$$

Since $\Omega_{R[X,Y]/R}$ is a free $R[X, Y]$-module of rank two with a basis dX, dY and $\mathrm{Im}(\theta) = N$ is the cyclic submodule generated by $F_X dX + F_Y dY$ where $F_X = \partial F/\partial X$ and $F_Y = \partial F/\partial Y$, it is enough to show that the ideal $(F_X, F_Y) = R[X, Y]$.

Suppose $\mathfrak{N}$ is a maximal ideal of $R[X, Y]$ such that $(F_X, F_Y) \subset \mathfrak{N}$. Let $\mathfrak{N} \cap R = \mathfrak{P}$. Then replacing R by $R_{\mathfrak{P}}$ and $\mathfrak{N}$ by $\mathfrak{N}_{\mathfrak{P}}$ we can assume that R is a local ring with the maximal ideal m, $\mathfrak{N}$ is a maximal ideal of $R[X, Y]$ with $\mathfrak{N} \cap R = m$ and $(F_X, F_Y) \subset \mathfrak{N}$. But then, since F is a residual variable, we have $R[X, Y] = (F_X, F_Y) + mR[X, Y] \subset \mathfrak{N}$ which is absurd. Hence $(F_X, F_Y) = R[X, Y]$.

Lemma 3.5. Let R be a noetherian ring such that no prime integer is a zero-divisor in R. Let $F \in R[X, Y]$ be such that $R[X, Y]/(F) = R[Z]$ as R-algebras. Then F is a residual variable.

Proof. Let $\mathfrak{P}$ be a prime ideal of R and let $\bar{F}$ denote the image of F in $k(\mathfrak{P})[X, Y]$. Then $k(\mathfrak{P})[X, Y]/(\bar{F}) = k(\mathfrak{P})[Z]$.

If $ht\ \mathfrak{P} = 0$, then since no prime integer is a zero-divisor in R, $k(\mathfrak{P})$ is a field of characteristic zero. Therefore by the Abhyankar-Moh epimorphism theorem ([2], Theorem 1.2) $k(\mathfrak{P})[X, Y] = k(\mathfrak{P})[\bar{F}]^{[1]}$.

If $ht\ \mathfrak{P} > 0$ then there exists a discrete valuation ring V of characteristic zero with the uniformizing parameter π and a ring homomorphism $\alpha: R \to V$ such that $\alpha^{-1}((\pi)) = \mathfrak{P}$ and the field extension $k(\mathfrak{P}) \to V/(\pi)$ (induced by α) is algebraic.

Let $\tilde{F}$ denote the image (through α) of F in $V[X, Y]$. Then $V[X, Y]/(\tilde{F}) = V[Z]$. Therefore by ([6], Theorem 2.6.2) $V[X, Y] = V[\tilde{F}]^{[1]}$ and hence $V/(\pi)[X, Y] = V/(\pi)[\bar{\tilde{F}}]^{[1]}$ where $\bar{\tilde{F}}$ is the image of $\tilde{F}$ in $V/(\pi)[X, Y]$.

Since we have the following commutative diagram of rings

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & V \\ \downarrow & & \downarrow \\ k(\mathfrak{P}) & \longrightarrow & V/(\pi) \end{array}$$

and $V/(\pi)$ is algebraic over $k(\mathfrak{P})$, by ([4], Proposition 1.16) $k(\mathfrak{P})[X, Y] = k(\mathfrak{P})[\bar{F}]^{[1]}$.

Thus we prove that F is a residual variable.

PROPOSITION 3.6.

Let R be a ring and I be an ideal of $R^{[n]}$ such that $R^{[n]}/I \approx R^{[n-1]}$ as R-algebras. Then I is a projective $R^{[n]}$-module of (constant) rank 1. Moreover if there exists a projective R-module L of rank 1 such that $L \otimes_R R^{[n]} \approx I$ as $R^{[n]}$-modules then I is a free $R^{[n]}$-module of rank 1 i.e. I is a principal ideal (necessarily generated by a non-zero-divisor of $R^{[n]}$).

Proof. It is easy to see that under the hypothesis of the proposition there exists a subring R' of R which is finitely generated over the ring of integers and an ideal I' of $R'^{[n]}$ such that $R'^{[n]}/I' \approx R'^{[n-1]}$ and $I = I'R^{[n]} \approx I' \otimes_{R'} R = I' \otimes_{R'^{[n]}} R^{[n]}$. Therefore for proving

the first part of the proposition we can assume without loss of generality that R is noetherian of finite Krull dimension.

We prove the result by induction on dim R.

Let dim $R = 0$. By Lemma 2.3 we can assume that R is reduced. But then R is a finite product of fields and hence, since $R^{[n]}/I \approx R^{[n-1]}$, I is a principal ideal (of height 1) generated by a non-zero-divisor. Therefore I is a free $R^{[n]}$-module of rank 1.

Now we assume that dim $R > 0$. Again by Lemma 2.3 we can assume that R is reduced. Let S be the set of non-zero-divisors of R. Then R_S is a finite product of fields and as before we conclude that I_S is a free $R_S^{[n]}$-module of rank 1. Therefore $\exists\, s \in S$ such that I_s is a free $R_s^{[n]}$-module of rank 1. We may assume that s is a nonunit of R.

Since $I \cap sR^{[n]} = sI$, $I/sI \approx I + sR^{[n]}/sR^{[n]}$ as $R/(s)^{[n]}$-modules. Therefore, since $R^{[n]}/(I + sR^{[n]}) \approx R/(s)^{[n-1]}$ and dim $R/(s) <$ dim R, by the induction hypothesis I/sI is a projective $R/(s)^{[n]}$-module. Since s is a non-zero-divisor of R, $I \subset R^{[n]}$ and I_s (resp. I/sI) is a projective $R_s^{[n]}$-module (resp. $R/(s)^{[n]}$-module) of rank 1, by Lemma 2.1 I is a projective $R^{[n]}$-module of (constant) rank 1.

Now assume that there exists a projective R-module L of rank 1 such that $L \otimes_R R^{[n]} \approx I$ as $R^{[n]}$-modules.

Since $R^{[n]}/I \approx R^{[n-1]}$ as R-algebras, we get the following right exact sequence of $R^{[n-1]}$-modules:

$$I/I^2 \to \Omega_{R^{[n]}/R}/I\Omega_{R^{[n]}/R} \to \Omega_{R^{[n-1]}/R} \to 0.$$

Since, for non-negative integer l, $\Omega_{R^{[l]}/R}$ is a free $R^{[l]}$-module of rank l and I/I^2 is a projective $R^{[n-1]}$-module (as I is projective over $R^{[n]}$ of rank 1) of rank 1 we see that the above sequence is also left exact and

$$\Omega_{R^{[n]}/R}/I\Omega_{R^{[n]}/R} \approx \Omega_{R^{[n-1]}/R} \oplus I/I^2.$$

Thus I/I^2 is a stably free $R^{[n-1]}$-module of rank 1 and therefore I/I^2 is free over $R^{[n-1]}$ of rank 1.

Let θ: $R^{[n-1]} \to R$ be a surjective R-algebra homomorphism. Then composite map

$$R \to R^{[n]} \twoheadrightarrow R^{[n]}/I \approx R^{[n-1]} \xrightarrow{\theta} R$$

is the identity automorphism of R.

Since $L \otimes_R R^{[n]} \approx I$, we get

$$\begin{aligned} L &= L \underset{R}{\otimes} R^{[n]} \underset{R^{[n]}}{\otimes} R^{[n-1]} \underset{R^{[n-1]}}{\otimes} R \approx I \underset{R^{[n]}}{\otimes} R^{[n-1]} \underset{R^{[n-1]}}{\otimes} R \\ &= I/I^2 \underset{R^{[n-1]}}{\otimes} R. \end{aligned}$$

But I/I^2 is a free $R^{[n-1]}$-module of rank 1. Hence L is a free R-module of rank 1 and therefore I is a free $R^{[n]}$-module of rank 1 i.e. I is a principal ideal.

Thus the proof of Proposition 3.6 is complete.

Now we prove Theorem A.

Theorem 3.7. *Let R be a ring such that R_{red} is seminormal and no prime integer is a zero-divisor in R_{red}. Let I be an ideal of $R[X, Y]$ such that $R[X, Y]/I = R[Z]$ (as R-algebras). Then I is a principal ideal say generated by F and $R[X, Y] = R[F]^{[1]}$.*

Proof. Since R_{red} is seminormal by ([7], Theorem 6.1) $\text{Pic}(R) = \text{Pic}(R^{[n]})$ for every n. Therefore by Proposition 3.6 I is a principal ideal say generated by F.

Let $\mathfrak{G}$ be the nilradical of R and let $\bar{F}$ be the image of F in $R/\mathfrak{G}[X, Y]$. If $R/\mathfrak{G}[X, Y] = R/\mathfrak{G}[\bar{F}]^{[1]}$ then it is easy to see that $R[X, Y] = R[F]^{[1]}$. Therefore we can assume that R is reduced. It is also easy to see that there exists a subring S of R which is finitely generated over the ring of integers such that $F \in S[X, Y]$ and $S[X, Y]/(F) = S[Z]$ as S-algebras. Note that S is a noetherian ring of finite Krull dimension.

Since $S \hookrightarrow R$ and R is reduced, by the hypothesis of the theorem, no prime integer is a zero-divisor in S. Therefore F is a residual variable in $S[X, Y]$ by Lemma 3.5. Hence $\Omega_{S[X,Y]/S[F]}$ is a free $S[X, Y]$-module of rank one by Lemma 3.4. Moreover by Lemma 3.3, for every prime ideal $\mathfrak{P}'$ of $S[F]$, $k(\mathfrak{P}') \otimes_{S[F]} S[X, Y] = k(\mathfrak{P}')^{[1]}$. $S[X, Y]$ is a (finitely generated) flat $S[F]$-algebra by Lemma 3.2. Therefore by ([3], Lemma 3.3) there exists a positive integer m such that $S[X, Y]^{[m]} = S[F]^{[m+1]}$.

Now $S[X, Y]^{[m]} = S[F]^{[m+1]}$ implies that $R[X, Y]^{[m]} = R[F]^{[m+1]}$. Since R is seminormal (we have assumed R to be reduced) by ([5], Theorem 2.6) $R[X, Y] = R[F]^{[1]}$.

Thus the proof of Theorem 3.7 is complete.

The following example shows that if R_{red} is not seminormal then $R[X, Y]/I = R[Z]$ need not imply that I is principal.

Example 3.8. Let k be a field of characteristic zero and let $\tilde{R} = k[[t]]$: a power series in one variable over k. Let $R = k[[t^2, t^3]]$, considered as a subring of $\tilde{R}$. It is obvious that $\tilde{R}$ is the normalization of R and R is not seminormal.

Let α: $R[X, Y] \to R[Z]$ be the R-algebra homomorphism defined as: $\alpha(X) = Z + t^3Z^2$ and $\alpha(Y) = t^2Z$. Let $I = \ker \alpha$. Then

(1) α is surjective

(2) I is not a principal ideal of $R[X, Y]$.

Proof. Since $\alpha(X - t^3X^2 + t^2XY^2 + Y^3) = Z$, α is surjective.

Let $\tilde{\alpha}$: $\tilde{R}[X, Y] \to \tilde{R}[Z]$ be the $\tilde{R}$-algebra homomorphism such that $\tilde{\alpha}(X) = \alpha(X) = Z + t^3Z^2$ and $\tilde{\alpha}(Y) = \alpha(Y) = t^2Z$. Let $\tilde{I} = \ker \tilde{\alpha}$. Then $\tilde{I}$ is a principal prime ideal of $\tilde{R}[X, Y]$ generated by $F(X, Y) = t^2X - Y - tY^2$. Moreover $\tilde{I} = I\tilde{R}[X, Y] (= I \otimes_R \tilde{R})$.

If I is a principal ideal of $R[X, Y]$ say generated by H then $H = uF$ $(= u(t^2X - Y - tY^2))$ where u is a unit in R and $ut \in R$ i.e. $t \in R$ which is a contradiction.

Thus we prove that I cannot be principal.

We conclude this section with the proof of Theorem B.

Theorem 3.9. *Let R be a ring containing a field k of characteristic zero. Let $F \in R[X, Y]$ such that $R[X, Y]/(F) = R[Z]$ as R-algebras. Then $R[X, Y] = R[F]^{[1]}$.*

Proof. As in Theorem 3.7, we can assume that R is reduced and R contains a noetherian subring S of finite Krull dimension such that $F \in S[X, Y]$ and $S[X, Y]/(F) = S[Z]$ as S-algebras. Moreover we can assume that S contains k. Repeating the same arguments we see that there exists a positive integer m such that $S[X, Y]^{[m]} = S[F]^{m+1}$. Now since S contains k (a field of characteristic zero) by ([5], Theorem 2.8), $S[X, Y] = S[F]^{[1]}$. Hence $R[X, Y] = R[F]^{[1]}$.

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The Dolbeault-cohomology ring of a compact, even-dimensional lie group

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Abstract. The paper presents a classification of all homogeneous (integrable) complex structures on compact, connected lie groups of even dimension. Thereafter, using lie algebraic methods it proves theorems about the Dolbeault cohomology rings of these complex manifolds in the semisimple case and exhibits the dramatic variation of ring structure of the Dolbeault rings of groups of rank 2. Using some specific computations for $SO(9)$, it gives a counter-example to a long-standing conjecture about the Hodge-deRham (Frohlicher) spectral sequence.

Keywords. Compact lie group; complex manifold; lie algebra; cohomology; spectral sequence.

1. Introduction

There is an old observation of H Samelson[†] (see Wang [10]) that a compact (connected) even dimensional lie group G can be made into a complex manifold in such a way that all left (or right) translations are holomorphic. If G is non-abelian it does not admit any closed symplectic two-form and so these complex structures cannot be Kählerian. Therefore, in principle, the Dolbeault cohomology ring $H_{\bar{\partial}}(G)$ can be quite different from the deRham cohomology $H^*_{DR}(G;\mathbb{C})$: and we shall show that this is indeed the case. In fact, one of the pcincipal results of this paper is that the difference can be so substantial that the Hodge-deRham spectral sequence for suitable complex structures on $SO(9)$ does *not* collapse at E_2: this gives the first counter-examples to a folk-lore conjecture concerning this spectral sequence. Another surprise is that the analogues of the Betti-numbers,

$$b_r(G) = \sum_{p+q=r} \dim_{\mathbb{C}} H^{p,q}(G)$$

can jump within a connected component of moduli.

We begin the paper by giving a complete description of the moduli of left-invariant, integrable complex structures on an arbitrary G: although Wang [10] had shown the existence of real lines in this space, he did not describe it any further. A simple

[†] Samelson's paper appeared in *Portugaliae Math.* which is perhaps understandably not available in Indian libraries.

computation of the Kodaira-Spencer infinitesimal deformation space shows that the moduli of left invariant structures has the "correct" dimension. The real work of computing $H_{\bar{\partial}}(G)$ beings in § 3, where we use Bott's lie-algebraic description [3] as a starting point for our computations of the ring structure in the semi-simple case. Section 4 gives a "uniform" description of $H_{\bar{\partial}}(G)$ for all simple groups of rank 2; here we already see the jumping of $b_4(G)$ but there are no surprises in the associated Hodge-deRham sequence. In § 5, we give a finite-dimensional "Koszul-type" complex to compute $H_{\bar{\partial}}(G)$ for arbitrary semi-simple G which shows that this ring depends only on the modulus and the polynomial invariants of the Weyl group on the maximal torus; using this description we show that Borel's tri-graded spectral sequence [2] for the *holomorphic* bundle $T \to G \to G/T$ collpases at E_3. In § 6 we discuss the moduli for $SO(9)$ and give the details of the computation of $H_{\bar{\partial}}(SO(9))$ for various complex structures on $SO(9)$, specifying in detail those which give counter-examples to the degeneration conjecture for the Hodge-deRham spectral sequence. The last section is concerned with the Atiyah–Chern classes [1] of the holomorphic tangent bundle of G, and shows that these too can vary with moduli.

The results of § 2 and § 5 imply at once that the moduli spaces of LICS on $SO(4k+1)$ and $Sp(2k)$ are the same, and under an appropriate identification the corresponding Dolbeault rings are isomorphic. Thus our computations in § 6 with $SO(9)$ carry over at once to $Sp(4)$: probably similar computations with $SO(8)$ and $SU(5)$ will give further counter-examples to the HdR conjecture, but the algebraic calculations for $SU(5)$ are more elaborate. (After this paper was submitted, the author learned of an announcement by Cordero, Fernandez and Gray "La Suite Spectrale de Fröhlicher et les nilvarietes Compacts, Complexes" in the *Comptes Rendus* vol. 305, Serie I, M. 753–756 (1987) which gives counter examples to the HdR conjecture of a quite different sort from our examples.)

At the end of § 5, we make a conjecture about another cochain complex to compute $H_{\bar{\partial}}(G)$ and produce some evidence in its favour. If true, this conjecture would substantially simplify the computations of these rings, and lead to further counter-examples for the Hodge-deRham conjecture. It would also facilitate the computation of variations of the Hodge structure for $H_{\bar{\partial}}(G)$.

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2. Complex structures on G

Let G be a compact, connected even-dimensional lie group. In this section we will give an explicit description of the *moduli space* of left-invariant, integrable complex structures on G, proving inter-alia that this space has positive dimension; and at the end of § 3 we will see that this dimension agrees with the dimension of $H^1(G, \Theta_G)$, the Kodaira–Spencer tangent space to the deformation space of all integrable complex structures (in a component containing a left-invariant structure). The positive-dimensionality of the moduli space is essentially proved in Wang [10] (Theorem C) but there is no explicit description of the space there and apparently no systematic discussion in the literature.

To facilitate matters we give an "infinitesimal" definition of a left-invariant

integrable complex structure: by the well-known Newlander–Nirenberg theorem (or actually simply by Frobenius integrability-using a real analytic structure on G) this is equivalent to the more naive definition in terms of systems of holomorphic charts for which left-translations are holomorphic maps. Let $\mathfrak{g}$ = the (real) lie-algebra of left invariant vector fields on G.

DEFINITION 2.1

A left-invariant, integrable complex structure (LICS) on G is a real linear map

$$J:\mathfrak{g}\to\mathfrak{g}$$

satisfying the following two conditions

(i) $J^2 = 1$

(ii) for $X, Y\in\mathfrak{g}, [JX,JY] = [X,Y] + J([JX,Y] + [X,JY])$.

We define $m_1(G)$ to be the closed subspace of $\mathrm{End}_{\mathbb{R}}(\mathfrak{g})$ of such maps: the automorphism group $\mathrm{Aut}(G)$ acts on $m_1(G)$ by conjugation, and we put $m_2(G) = m_1(G)/\mathrm{Aut}(G)$, with its quotient topology. The main result of this section is that $m_2(G)$ is "essentially" (up to the action of a discrete group) the set of linear maps of square -1 on a maximal toral subalgebra of $\mathfrak{g}$ (see 2.7.1 below).

Now it is standard linear algebra that on a real, even-dimensional vector space V, the linear endomorphisms $J:V\to V$ of square -1_V are in $1-1$ correspondence with complex subspaces $W\subseteq V\otimes\mathbb{C} = V^c$ of half the dimension which intersect the given real form V in $\{0\}$. Thus an LICS on G gives a complex subspace $s\subseteq\mathfrak{g}^c$ being the $+i$-eigenspace of J: and the condition (ii) says precisely that s (and also $\bar{s}$) is a complex lie subalgebra of $\mathfrak{g}^c$.

PROPOSITION 2.2

The points of $m_1(G)$ *are in* $1-1$ *correspondence with complex lie subalgebras* $s\subseteq\mathfrak{g}\otimes\mathbb{C} = \mathfrak{g}^c$ *such that*

(a) $s\cap\mathfrak{g} = \{0\}$ (b) $\dim_{\mathbb{C}} s = \frac{1}{2}\dim_{\mathbb{C}}\mathfrak{g}^c$. □

For later use we recall that if V is equipped with a positive definite inner product $\langle,\rangle$ then the J's which are orthogonal correspond to subspaces $W\subseteq V^c$ which are totally isotropic for the complex bilinear extension of $\langle,\rangle$ to V^c. Thus an orthogonal LICS on (with respect to the Killing form) will correspond to a subalgebra $s\subseteq\mathfrak{g}^c$ which is maximal isotropic for the Killing form of $\mathfrak{g}^c$.

A subalgebra of $\mathfrak{g}^c$ satisfying the conditions of 1.2 will be called a *Samelson subalgebra*, because Samelson first constructed particular examples by using an explicit Weyl basis for $\mathfrak{g}^c$. Samelson's construction gave subalgebras which were isotropic for the Killing form, and hence solvable. We shall see that *all* Samelson subalgebras are solvable, although not necessarily totally isotropic. We now describe the structure of a general Samelson subalgebra.

Let $\tilde{G}^c$ be the connected, simply-connected complex lie group for $\mathfrak{g}^c$ and $\tilde{G}, S$ the closed connected subgroups of $\mathfrak{g}, s\subseteq\mathfrak{g}^c$.

PROPOSITION 2.3

The multiplication map $\tilde{\mu}:\tilde{G}\times S\to\tilde{G}^c$ *is a* C^∞ *diffeomorphism.*

Proof. Since $\mathfrak{g}'$ and s are complementary subalgebras of $\mathfrak{g}^c$, an elementary lemma attributed to Harish–Chandra (Helgason [4] lemma 5.2) asserts that the differential of $\tilde{\mu}$ is of maximal rank, and so $\tilde{\mu}$ is an open mapping. We will prove that $\tilde{\mu}$ is surjective, and injective: thus $\tilde{\mu}$ will be a diffeomorphism.

There is a discrete, central subgroup $\pi\subseteq\tilde{G}$ such that $\pi\backslash\tilde{G}=K$ is a compact group and π is a discrete subgroup of $\tilde{G}^c$ as well, so there is a diagram

$$\begin{array}{ccc}\tilde{G}\times S & \xrightarrow{\tilde{\mu}} & \tilde{G}^c \\ \downarrow & & \downarrow \\ K\times S & \xrightarrow{\rho} & X=\pi\backslash\tilde{G}^c\end{array} \tag{2.3.1}$$

the vertical maps being quotients by π. Now for every $k\in K$, the image $\rho(\{k\}\times S)$ is closed in X and hence the compactness of K implies that ρ has closed image. On the other hand, since $\tilde{\mu}$ is an open mapping so is ρ: and since X is connected, ρ is surjective. Then, from the diagram above it follows that $\tilde{\mu}$ is surjective as well.

Next consider the closed subgroup $H=\tilde{G}\cap S\subseteq\tilde{G}^c$. It is discrete because its lie algebra is $\mathfrak{g}\cap s=\{0\}$. H acts freely on the product $\tilde{G}\times S$ by the rule,

$$h:(g,s)=(gh^{-1},hs)$$

and the orbits of H are precisely the fibres of $\tilde{\mu}$. Since H is discrete and we have already established $\tilde{\mu}$ to be an open surjection, $\tilde{\mu}$ is a principal H-covering of $\tilde{G}^c$: but $\tilde{G}^c$ is simply-connected and $\tilde{G}\times S$ is connected so $H=\{e\}$ and $\tilde{\mu}$ is injective as well. //

This proposition has several important consequences. First of all, since $\tilde{G}\times S\xrightarrow{\sim}\tilde{G}^c$, $\tilde{G}$ and S are both simply-connected. Thus $\tilde{G}$ is the universal cover of our group G and there is a discrete central subgroup $\Gamma\subseteq\tilde{G}$ such that $\tilde{G}/\Gamma=G$. Then it is known that Γ is (discrete and) central in $\tilde{G}^c$ as well, so there is a complex lie group $G^c=\tilde{G}^c/\Gamma$ such that the multiplication map $\mu:G\times S\to G^c$ induced by $\tilde{\mu}$ is a diffeomorphism. In particular, taking quotients by the right-action of S,

$$G\xrightarrow{\sim}G^c/S,$$

which shows that G is a complex homogeneous space of G^c, the complex structure on G being the one induced by the LICS corresponding to S. Even more, there is a principal holomorphic S-bundle

$$S\to G^c\xrightarrow{f}G \tag{2.3.2}$$

defining the given LICS. Thus various natural holomorphic vector bundles on G are the homogeneous vector bundles associated to the corresponding complex S-modules. For example, the holomorphic tangent bundle of G is associated to the ad(S)-module

$\mathfrak{g}^c/s$, and the holomorphic bundle of p-covectors is associated to the ad(S)-module $\Lambda^p(\mathfrak{g}^c/s)^*$.

PROPOSITION 2.4

S is solvable.

Proof. G being a maximal compact subgroup of G^c, the inclusion $i: G \to G^c$ is a homotopy equivalence and a C^∞-section for the map f; hence S is contractible.

Now a *complex* contractible group must be solvable. In the Levi decomposition of S, the connected component of the radical is simply-connected and solvable as a real group, hence contractible. Thus the semi-simple factor of S is contractible. The semi-simple part being a complex semi-simple group, and contractible, it follows from Weyl's unitarian trick that it must reduce to the identity[†]. //

I do not know of a purely lie-algebraic argument to show that a Samelson algebra is solvable. However, a real contractible group need not be solvable, the standard example being the universal cover $SL(2, \tilde{\mathbb{R}})$.

Since s is solvable, it is contained in a Borel (= maximal solvable) subalgebra of $\mathfrak{g}^c$, and our next result is that this Borel subalgebra is uniquely determined by s. This result will then lead easily to our descriptions of $m_1(G)$ and $m_2(G)$.

PROPOSITION 2.5

There is a unique Borel subalgebra containing a given Samelson subalgebra s.

Proof. Every Borel subalgebra is the normalizer of its commutator ideal: hence for any $\mathfrak{b}$ containing s, it suffices to show $[s, s] = [\mathfrak{b}, \mathfrak{b}]$. Fix a $\mathfrak{b}$ containing s, and choose its Cartan subalgebra to be $\mathfrak{h} \equiv$ complex-linear span of $\mathfrak{b} \cap \mathfrak{g}$: let $2k = \dim_{\mathbb{C}} \mathfrak{h}$, $2k + n = \dim_{\mathbb{C}} \mathfrak{b}$ so that $k + n = \dim_{\mathbb{C}} s$ and $n = \dim_{\mathbb{C}} [\mathfrak{b}, \mathfrak{b}]$. Since $[s, s] \subseteq [\mathfrak{b}, \mathfrak{b}]$, it will suffice to show that $\dim_{\mathbb{C}} [s, s] \geqslant n$.

The subalgebra $\mathfrak{a} = s \cap \mathfrak{h}$ has no non-zero real points, so $\dim_{\mathbb{C}} \mathfrak{a} \leqslant k$. On the other hand $\mathfrak{h}/(s \cap \mathfrak{h})$ injects into the vector space $\mathfrak{b}/s$, so by a dimension count, $\dim_{\mathbb{C}} \mathfrak{a} = k$. Now $\mathfrak{a}$ acts reductively on $\mathfrak{b}$, and we claim that its centralizer in $\mathfrak{b}$ (= space of vectors of weight 0 for $\mathfrak{a}$) is $\mathfrak{h}$. Indeed, putting $\bar{\mathfrak{a}} =$ conjugate of $\mathfrak{a}$ with respect to the real form $\mathfrak{b} \cap \mathfrak{g}$, we have $\mathfrak{h} = \mathfrak{a} \oplus \bar{\mathfrak{a}}$: and since the roots for $(\mathfrak{b}, \mathfrak{h})$ are purely imaginary on the given real form, none of them can vanish on $\mathfrak{a}$.

The facts that $\mathfrak{a} \subseteq s$ and $s \cap \mathfrak{g} = \{0\}$ easily imply that $s \cap \bar{\mathfrak{a}} = \{0\}$ hence the centralizer of $\mathfrak{a}$ in s is $\mathfrak{a}$ itself. Thus decomposing s an ad$(\mathfrak{a})$ module.

$$s = \mathfrak{a} \oplus \Sigma s_\lambda,$$

where λ runs over a (finite) set of non-zero linear forms on $\mathfrak{a}$, and $s_\lambda =$ "λ-eigenspace" of $\mathfrak{a}$ in s. Notice that $\dim_{\mathbb{C}}(\Sigma s_\lambda) = n$. Since $\lambda \neq 0$, we have $s_\lambda = [\mathfrak{a}, s_\lambda] \subseteq [s, s]$ and so $\Sigma s_\lambda \subseteq [s, s]$ which gives $\dim_{\mathbb{C}} [s, s] \geqslant n$. //

[†] In fact the same argument shows that if H is a complex lie group with $\pi_j(H) = 0$ for $0 \leqslant j \leqslant 3$, then H is solvable.

COROLLARY 2.5.1

Every Samelson subalgebra $s \subseteq \mathfrak{g}^c$ *is even as a canonically split extension*

$$0 \to \mathfrak{n} \to s \to \mathfrak{a} \to 0$$

where $\mathfrak{n} \subseteq \mathfrak{g}^c$ *is a maximal nilpotent subalgebra, and* $\mathfrak{a}$ *is a reductive abelian subalgebra of complex dimension* $k = \frac{1}{2}(\text{rank } \mathfrak{g}^c)$ *which intersects* $\mathfrak{g}$ *in* $\{0\}$.

Proof. This is just a rewording of the proof of 2.5. //

COROLLARY 2.5.2

$m_1(G)$ *is in* $1-1$ *correspondence with the set of pairs*

$$\{(\mathfrak{b}, J)\},$$

where $\mathfrak{b} \subseteq \mathfrak{g}^c$ *is a Borel subalgebra and* $J \in \text{End}_{\mathbb{R}}(\mathfrak{b} \cap \mathfrak{g})$ *is a linear map of square* -1. *We have thus encoded the integrability condition on* J *into the structure of* $\mathfrak{g}^c$.

Fix a topological space X and a real vector bundle $\mathbb{R}^{2k} \to V \to X$ of rank $2k$ over X. Define $\mathcal{J}(\mathbb{R}^{2k})$ to be the subset of $\text{End}_{\mathbb{R}}(\mathbb{R}^{2k})$ of maps of square -1, and $\mathcal{J}(V) \to X$ to be the associated bundle over X: that is, $\mathcal{J}(V)$ is the subset of $\text{End}_{\mathbb{R}}(V)$ of fibre-wise linear maps of square -1.

Let $E \to G/T$ be the trivial real vector-bundle over G/T associated to the (trivial) adjoint action of T on its lie algebra t.

PROPOSITION 2.6

$m_1(G) \simeq \mathcal{J}(E)$.

Proof. The homogeneous space G^c/B is the variety of all Borel subalgebras of $\mathfrak{g}^c$, and is isomorphic to G/T. From 2.5.2 there is a natural map $m_1(G) \xrightarrow{f} G/T$ which lifts to $\mathcal{J}(E)$ as a $1-1$ onto map. The continuity of the lifted map is essentially a matter of linear algebra: the assignment

$$J(\varepsilon \text{End}_{\mathbb{R}}(\mathfrak{g})) \to s$$

is continuous, as are the assignments $s \to \mathfrak{b}$, $s \to \mathfrak{a}$, all for essentially standard reasons. //

Remarks 2.6.1.

(a) Since $E \to G/T$ is a trivial bundle,

$$\mathcal{J}(E) = \mathcal{J}(\mathbb{R}^{2k}) \times G/T = GL(2k, \mathbb{R})/GL(k, \mathbb{C}) \times G/T.$$

(b) Under the isomorphism in part (a) the orthogonal J's (equivalently isotropic s) fill up the space $O(2k)/U(k) \times G/T$.

As a consequence of 2.6.1 if G has more than 1 simple factor, the LICS on G (even the orthogonal ones) do not decompose as a product of LICS on the factors: in fact those that do decompose form a proper subvariety. This will show later on

that there is no obvious Künneth formula for $H_{\bar\partial}(G)$ in terms of its factors (see 5.6).

We can now describe $m_2(G) = m_1(G)/\mathrm{Aut}(G)$. Recall that the automorphism group $\mathrm{Aut}(G)$ is given as an extension

$$0 \to \bar G \to \mathrm{Aut}(G) \to F \to 0,$$

where the adjoint group $\bar G = G/(\text{centre})$ is the group of inner automorphisms, and F is a discrete group generated by the automorphisms of the abelian "factor" of G, the automorphisms of the Dynkin diagrams of the simple factors, and permutations among isomorphic simple factors. The normality of $\bar G$ in $\mathrm{Aut}(G)$ implies that we may form the quotient

$$m_1(G) = m_1(G)/\bar G$$

and then take the quotient by F to obtain $m_2(G)$.

PROPOSITION 2.7

$m'(G) \simeq \mathscr{J}(\mathbb{R}^{2k}) = GL(2k, \mathbb{R})/GL(k, \mathbb{C})$.

Proof. In terms of 2.5.2., the action of G (equivalently $\bar G$) on $m_1(G)$ is given by $g.(\mathbf{b}, J) = (\mathrm{ad}(g)\mathbf{b}, \mathrm{ad}(g)J)$. The isomorphism $G^c/B \simeq G/T$ shows that the adjoint action of G is transitive on Borel subalgebras. Hence, if we fix one, say $\mathbf{b}_0 \subseteq \mathfrak{g}^c$ then under the action of G every pair $(\mathbf{b}, J)$ is equivalent to some $(\mathbf{b}_0, J)$. Thus there is a surjective map $\mathscr{J}(\mathbf{b}_0 \cap \mathfrak{g}) \to m'(G)$ because on $\mathbf{b}_0 \cap \mathfrak{g}$ we can have J be arbitrary. The stabilizer of $\mathbf{b}_0$ in G is $G \cap B_0 = T_0$ which acts trivially on $\mathrm{b}_0 \cap \mathrm{g} = t_0$ and hence also on $\mathscr{J}(\mathbf{b}_0 \cap \mathfrak{g}) = \mathscr{J}(\mathbb{R}^{2k})$. //

COROLLARY 2.7.1

$m_2(G) = \mathscr{J}(\mathbb{R}^{2k})/F = F \backslash GL(2k, \mathbb{R})/GL(k, \mathbb{C})$.

Proof. We only have to identify the action of F on $\mathscr{J}(\mathbb{R}^{2k})$. Now F comes equipped with a representation on $GL(\mathfrak{h}, \mathbb{R}) = GL(2k, \mathbb{R})$ being the product of $GL(?, Z)$ on the abelian factor and the permutation representations of the simple roots specified in the Dynkin diagrams—and this gives the isomorphism.

We illustrate the case of simple groups of rank 2.

Example. For $k = 1$, $GL(2, \mathbb{R})/GL(1, \mathbb{C}) \simeq \mathscr{H}_+ \cup \mathscr{H}_-$, where $\mathscr{H}_+$ are the upper and lower half planes in $\mathbb{C}$. Hence

(a) if $G = G_2$ or $\mathrm{Spin}(5) = \mathrm{Sp}(2)$ then

$$m_2(G) = \mathscr{H}_+ \cup \mathscr{H}_- \text{ because } F = \{e\} \text{ in these cases.}$$

(b) $m_2(SU(3)) \simeq \mathscr{H}$ because $F = Z/2$ and one simply has to check that it acts by complex conjugation under the identifications made above.

Note that in this case $O(2)/U(1) \simeq \{\pm l\}$ as a subset of $\mathscr{H}_+ \cup \mathscr{H}_-$ and hence there is a *unique* orthogonal (equiv. isotropic) LICS in each component of $m_2(G)$.

We end by recording some miscellaneous facts which are required later on.

2.8 Remarks

(a) Let $\mathbf{b}$ be the unique Borel containing s. Since $s \supseteq [\mathbf{b}, \mathbf{b}]$ it follows that s is an *ideal* in $\mathbf{b}$ and hence at the group level, that S is a closed, complex normal subgroup of B and B/S is a complex abelian group.
(b) Hence there is a principal holomorphic fibre bundle

$$B/S \to G^c/S \to G^c/B.$$

Now $G^c/S \simeq G$ is compact and so B/S is a complex torus. Indeed, the bundle just given can be identified with

$$T \to G \to G/T$$

showing that every LICS on G makes the above C^∞-bundle into a holomorphic one.

3. Preliminary reductions

We begin this section by showing that $H_{\bar\partial}(G)$ is independent of finite-covering: thus for computations it is sufficient to consider the cases in which G is simply-connected, or the product of a simply-connected group with a torus. Thereafter we shall restrict attention to the case that G is simply-connected and even-dimensional because Bott's lie-algebra formulae can be applied most directly to this case. The remaining cases have to be handled by fibering the group over a simply-connected, homogeneous complex manifold, and this is better discussed in the context of the $\bar\partial$-cohomology of general (non-Kähler) compact, complex, homogeneous spaces which we plan to take up elsewhere.

The independence of $H_{\bar\partial}(\cdot)$ of coverings is certainly suggested by Bott's lie-algebra formulae: however it is not implied by them since Bott works entirely in the framework of simply-connected groups. Thus one either has to redo Bott's proofs for the general case, or else to adopt the *ad hoc* procedure given below. In any case, Bott's formulae are not valid if $\pi_1(G)$ is infinite as one can see easily from examples.

So suppose G is a compact, connected even-dimensional group and that $\tilde G \xrightarrow{\pi} G$ is a finite covering with group Γ. Then since an LICS is determined by lie-algebraic data, $m_2(G) = m_2(\tilde G)$ and π is a holomorphic map between the corresponding complex manifolds. Therefore there is an induced map

$$\pi^* H_{\bar\partial}(G) \to H_{\bar\partial}(\tilde G)$$

which we will now prove is an isomorphism.

From 2.8 (b) we have maps of holomorphic principal bundles

$$\begin{array}{ccccc} \tilde T & \longrightarrow & \tilde G & \longrightarrow^{\tilde G} & /\tilde T \\ \downarrow & & \downarrow \pi & & \downarrow \| \\ T & \longrightarrow & G & \longrightarrow^{G} & /T \end{array} \qquad \begin{matrix} \\ \\ (3.1) \\ \\ (3.1') \end{matrix}$$

with π having kernel Γ in both verticals. Thus in complex cohomology, $\pi^*: H_{\bar\partial}(T) \to H_{\bar\partial}(\tilde T)$ is an isomorphism, since the (dual) lattice of T has finite index in that that of $\tilde T$. Therefore the map π induces an isomorphism at the E_2-level of Borel's spectral sequence [2] for the Dolbeault cohomology of (3.1) and (3.1′), and hence an isomorphism on the E_∞-levels. Thus π^* is an isomorphism in $\bar\partial$-cohomology as claimed.

Henceforth, G is compact, simply-connected of rank $2k$. We now recall Bott's description of $H_{\bar\partial}(G)$ in terms of lie-algebra cohomology and use it to compute the two lower edges of the Hodge diamond, $H^{0,p}(G)$ and $H^{p,0}(G)$. Serre duality then gives the upper edges as well, and we shall tackle the inner portion in the next two sections.

Let M be a compact, simply-connected complex manifold and G^c a complex reductive group acting holomorphically and transitively on M. Fixing a point $m_0 \in M$ and its isotropy subgroup $U \subseteq G^c$, we have a holomorphic principal U-bundle

$$U \to G^c \xrightarrow{\pi} M$$

and hence for any finite-dimensional holomorphic U-module V, a homogeneous holomorphic vector bundle

$$V \to E_V \to M.$$

Let $\mathcal{O}_M(V)$ denote the sheaf of holomorphic sections of this bundle. Bott [3] decomposes the cohomology spaces $H^p(M, \mathcal{O}_M(V))$ as modules over the maximal compact subgroup $G \subseteq G^c$: and expresses the pieces in terms of the lie-algebra cohomology of $G^c, U, U \cap G$. In our case $M = G$, $U = S$, and the homogeneous bundles are those associated to the sheaves of holomorphic p-forms Ω^p_G: namely, they are the bundles associated to the adjoint action of S on $\Lambda^p(\mathfrak{g}^c/s)^*$. Moreover, G acts trivially on $H^p(G, \Omega^q_G)$. With these simplifications, his Theorem II takes the following form:

Bott's Formula (3.2). $H^p(G, \Omega^q_G) \simeq H^p(s, \Lambda^q(\mathfrak{g}^c/s)^*)$.

It is standard yoga that these isomorphisms are compatible with the natural ring structures on $H^*(G, \Omega^*_G)$ and $H^*(s, \Lambda^*(\mathfrak{g}^c/s)^*)$.

We now adopt the notation $\mathfrak{m} = (\mathfrak{g}^c/s)^*$ for the ad(s)-module $(\mathfrak{g}^c/s)^*$ so that Bott's formula takes the form

$$H^p(G, \Omega^q_G) \simeq H^p(s, \Lambda^q(\mathfrak{m}))$$

and we start by recording the following simple structural fact about $\mathfrak{m}$.

PROPOSITION 2.3

There is an exact sequence of ad(s)*-modules*

$$0 \to \mathfrak{n} \to \mathfrak{m} \to \mathbb{C}^k \to 0,$$

where $\mathbb{C}^k$ *denotes the trivial s-module of* rank k.

Proof. Since the inclusion of $\mathfrak{b}$ in $\mathfrak{g}^c$ is a map of ad(s)-modules, there is an exact

sequence of ad(s)-modules

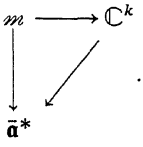
$$0 \to (\mathfrak{g}^c/\mathfrak{b})^* \to (\mathfrak{g}^c/s)^* \to (\mathfrak{b}/s)^* \to 0$$

and our claim amounts to proving the isomorphisms of ad(s)-modules

$$(\mathfrak{g}^c/\mathfrak{b})^* \simeq \mathfrak{n}, \quad (\mathfrak{b}/s)^* \simeq \mathbb{C}^k.$$

In fact since s is an ideal in $\mathfrak{b}$, the sequence above is an exact sequence of ad($\mathfrak{b}$)-modules and the isomorphisms claimed above are isomorphisms of ad($\mathfrak{b}$)-modules. Indeed the isomorphism $\mathfrak{b}/s \simeq \mathbb{C}^k$ is clear since $s \supseteq [\mathfrak{b}, \mathfrak{b}]$ (see 2.5) and hence the dual module is trivial as well. The other isomorphism is "well-known" from the structure theory of the complex semi-simple algebra $\mathfrak{g}^c$. //

Note that in the special case that s happens to be a totally-isotropic subspace for the Killing form (or some other G-invariant complex bilinear form on $\mathfrak{g}^c$) then $s \simeq m$, as ad(s) modules; and then the exact sequence of 3.3 is just the sequence giving the structure of s (2.5.1). The difference between s and m in general can be made precise as follows. Using the exact sequence in the proof above, it is easy to see that there is a diagram



In the isotropic case, $\bar{\mathfrak{a}}^* = \mathfrak{a}$ and the extension is then the one giving s.

PROPOSITION 3.4

$H^*(s{:}\Lambda^*(m)) \simeq H^*(\mathfrak{a}) \otimes H^*(\mathfrak{n}{:}\Lambda^*(m))^{\mathfrak{a}}$

Proof. Quite generally, let M be any finite-dimensional s-module on which $\mathfrak{a}$ acts reductively: and consider the Hochschild–Serre spectral sequence [6] for the split extension

$$0 \longrightarrow \mathfrak{n} \longrightarrow s \longrightarrow \mathfrak{a} \longrightarrow 0$$

abutting to $H^*(s{:}M)$. Then it is essentially theorem 13 of [6] that $H^*(s{:}M) \simeq H^*(\mathfrak{a}) \otimes H^*(\mathfrak{a}{:}M)^{\mathfrak{a}}$, the isomorphism being natural-so that if M is an s-algebra, then this is an isomorphism of graded algebras. The only difference is that the statement of their theorem 13 requires the quotient algebra ($\mathfrak{a}$ in our case) to be semi-simple: but a glance at this proof shows that all they really need is that it be reductive[†]. Hence the result follows by taking $M = \Lambda^*(m)$. //

In particular, we see that $H^*_{\delta}(G)$ is a free module over $H^*(\mathfrak{a})$; and hence to describe it as an algebra we only need to describe the subalgebra $H^*(\mathfrak{n}{:}\Lambda^*(m))^{\mathfrak{a}}$. We begin by

[†] It is worth mentioning that the Hochschild–Serre paper [6] was written before the term *reductive* was in common use in this context.

computing $H^p(\mathfrak{n}:\mathbb{C})^{\mathfrak{a}}$ and $H^0(\mathfrak{n}:\Lambda^q(\mathfrak{m}))^{\mathfrak{a}}$, which correspond to $H^{0,p}(G)$ and $H^{q,0}(G)$. The "inner part" of the Hodge diamond—i.e. $H^p(\mathfrak{n}:\Lambda^q(\mathfrak{m}))^{\mathfrak{a}}$ with $p, q \geqslant 1$—will be taken up in the next two sections.

Recall that our choice of Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{b}$ comes equipped with a real form $t = \mathfrak{h} \cap \mathfrak{g}$. Thus it makes sense to speak of a weight $\lambda:\mathfrak{h} \to \mathbb{C}$ being real or purely imaginary: i.e. $\lambda(t) \subseteq \mathbb{R}$ or $i\mathbb{R}$

Lemma 3.5. Suppose M is a finite sum of one-dimensional $\mathfrak{h}$-modules with real (or imaginary) weights. Then the invariant subspaces $M^{\mathfrak{a}}$ and $M^{\mathfrak{h}}$ coincide.

Proof. This is just another way of saying that if $\lambda:\mathfrak{h} \to \mathbb{C}$ is a real (or imaginary) functional, then $\lambda(\mathfrak{a}) = 0$ if and only if $\lambda = 0$; which is clear from $\mathfrak{h} = \mathfrak{a} + \bar{\mathfrak{a}}$ and $\mathfrak{a} \cap \bar{\mathfrak{a}} = \{0\}$. //

COROLLARY 3.5.1

For $p \geqslant 1$, $\Lambda^p(\mathfrak{n})^{\mathfrak{a}} = \Lambda^p(\mathfrak{n}_-)^{\mathfrak{a}} = 0$.

Proof. The two cases are parallel, since $\mathfrak{n}_-$ is dual to $\mathfrak{n}$ as an $\mathfrak{h}$-module. Now from the lemma,

$$\Lambda^p(\mathfrak{n})^{\mathfrak{a}} = \Lambda^p(\mathfrak{n})^{\mathfrak{h}}$$

since $\Lambda^p(\mathfrak{n})$ is a sum of 1-dimensional weight spaces (corresponding to p-fold sums of the positive roots): and since the weights are all positive, none of them can vanish on $\mathfrak{h}$. //

PROPOSITION 3.6

$H^{0,p}(G) \simeq H^p(\mathfrak{a}) = \Lambda^*(\mathfrak{a}^*)$.

Proof. From Bott's formula and the Hochschild–Serre decomposition

$$H^{0,p}(G) = H^p(s:\mathbb{C}) \simeq \sum_{r=0}^{p} H^r(\mathfrak{a}) \otimes H^{p-r}(\mathfrak{n}:\mathbb{C})^{\mathfrak{a}}$$

whence it sufficies to show that $H^t(\mathfrak{n}:\mathbb{C})^{\mathfrak{a}} = 0$ for $t \geqslant 1$. Now $\mathfrak{a}$ acts reductively on $\mathfrak{n}$, and therefore also on the cochain complex $(\Lambda^*(\mathfrak{n})^*, d)$ which computes $H^*(\mathfrak{n}:\mathbb{C})$: therefore

$$H^*(\mathfrak{n};\mathbb{C})^{\mathfrak{a}} = H^*(\Lambda^*(\mathfrak{n}^*)^{\mathfrak{a}}, d)$$

and the right-hand side is zero in positive degrees by 3.5. //

Notice that $H^p(\mathfrak{a}) \simeq H^{0,p}(T)$ where T is a "complex" maximal torus in G, as in 2.8. Our next result is that G has no holomorphic p-forms on it for $p \geqslant l$ (recall $\pi_1(G) = 0$): in § 5 we will give *two* shorter derivations of this result, but it seems worthwhile to have a direct "nuts-and-bolts" proof without recourse to any homological machinery.

To prepare the ground, if we fix $s, \mathfrak{b}, \mathfrak{h}$ as above, we can write $\mathfrak{g}^c$ as an $\mathrm{ad}(\mathfrak{h})$ module as

$$\mathfrak{g}^c = \mathfrak{h} \oplus \mathfrak{n} \oplus \mathfrak{n}',$$

where $\mathfrak{n}, \mathfrak{n}'$ are the nilpotent subalgebras corresponding to the positive and negative roots. (We choose generators $X_\alpha \in \mathfrak{n}_\alpha$ and $Y_\alpha \in \mathfrak{n}'_\alpha$ satisfying trace $(\mathrm{ad}(X_\alpha)\,\mathrm{ad}(Y_\alpha)) = 1$ and dual elements $\xi_\alpha \in \mathfrak{n}_\alpha^*, \eta_\alpha \in \mathfrak{n}_\alpha'^*$; and we note that under the duality isomorphisms of $\mathfrak{h}$-modules $\mathfrak{n}^* \simeq \mathfrak{n}', \mathfrak{n}'^* \simeq \mathfrak{n}$, ξ_α corresponds to Y_α and η_α to X_α.

The next result is incompatible with the Example worked out by Bott [3] (page 205) where it is claimed that

$$h^{1,0}(SU(3)) = \dim_{\mathbb{C}} H^{1,0}(SU(3)) = 1.$$

However, there seem to be several typographical misprints there; for example he states $h^{8,8}(SU(3)) = 1$, which is impossible since $\dim_{\mathbb{C}} SU(3) = \frac{1}{2}\dim_{\mathbb{C}} SU(3) = 4$. Compare our 4.5.

PROPOSITION 3.7

$H^{p,0}(G) = 0$ *for* $p \geqslant 1$.

Proof. From 3.2, $H^{p,0}(G) = H^0(s{:}\Lambda^p(\mathfrak{m})) = \Lambda^p(\mathfrak{m})^s$. Clearly we have $\Lambda^p(\mathfrak{m})^s \subseteq \Lambda^p(\mathfrak{m})^{\mathfrak{a}}$: and since $\mathfrak{a}$ acts reductively on $\mathfrak{m}$, we have $\mathfrak{m} \simeq \mathbb{C}^k \oplus \mathfrak{n}$ as $\mathfrak{a}$-modules where $\mathbb{C}^k \subseteq \mathfrak{m}$ is the subspace $\bar{\mathfrak{a}}^* \{\theta \in \mathfrak{h}^* \mid \theta(\mathfrak{a}) = 0\}$. Therefore as $\mathfrak{a}$-modules

$$\Lambda^p(\mathfrak{m}) \simeq \sum_{q=0}^{p} \Lambda^q(\mathbb{C}^k) \otimes \Lambda^{p-q}(\mathfrak{n})$$

and taking $\mathfrak{a}$-invariants and applying 3.4.1, $\Lambda^p(\mathfrak{m})^{\mathfrak{a}} = \Lambda^p(\mathbb{C}^k)$.

To show $\Lambda^p(\mathfrak{m})^s = 0$ for $p \geqslant 1$ it is enough to prove that there is no $\mathfrak{n}$-invariant element in the subspace $\Lambda^p(\mathfrak{m})^{\mathfrak{a}}$ of vectors of weight 0. Recall how $\mathfrak{n}$ acts on an element of $\Lambda^p(\mathbb{C}^k)$.

If $\theta_1, \ldots, \theta_k$ is a basis for $\bar{\mathfrak{a}}^*$ and $X_\alpha \in \mathfrak{n}$ is a root vector chosen as above then

$$X_\alpha \cdot \theta_j = -\langle \theta_j, \alpha \rangle \eta_\alpha \in \mathfrak{m},$$

so that the action of X_α on a monomial $\theta_{j_1} \wedge \cdots \wedge \theta_{j_p}$ is given by

$$X_\alpha \cdot (\theta_{j_1} \wedge \cdots \wedge \theta_{j_p}) = -\sum_{l=1}^{p} \langle \theta_{j_l}, \alpha \rangle (\theta_{j_1} \wedge \cdots \wedge \hat{\theta}_{j_l} \cdots \wedge \theta_{j_p}) \eta_\alpha.$$

Our claim is that for this action there is no non-zero linear combination $\sum_I c_I \theta_I$ invariant under all X_α (I being a multi-index of degree p).

The formula above giving the action of X_α on θ_I shows that $X_\alpha \cdot (\Sigma c_I \theta_I)$ will be a linear combination of monomials of the form $\theta_J \wedge \eta_\alpha$ where J is a multi-index of degree $p-1$. Hence it suffices to fix a *fixed* multi-index J_0 and to show that $\sum_{j \notin J_0} c_j \theta_{J_0 \cup \{j\}}$ cannot be $\mathfrak{n}$-invariant unless all the $c_j = 0$. Now

$$X_\alpha \cdot \Sigma c_j \theta_{J_0 \cup \{j\}} = -\left(\sum_j c_j \langle \theta_j, \alpha \rangle \theta_{J_0} \right) \eta_\alpha + \text{other lin.ind.terms}$$

so if this expansion is zero for all α then

$$\langle \Sigma c_j \theta_j, \alpha \rangle = 0 \quad \text{for all } \alpha,$$

which implies $\Sigma c_j \theta_j = 0$ whence $c_j = 0$ for all admissible j. //

Notice that in particular, G cannot be complex-parallelizable: for if it were, then the holomorphic cotangent bundle would be trivial, giving rise to global holomorphic one-forms.

Henceforth our major concern will be with the groups $H^*(\mathfrak{n}, \Lambda^*(\mathscr{m}))^{\mathfrak{a}}$ and related ones, so we set down the following convenient abbreviation. For any s-module, M we define the *invariant-cohomology of* M to be $H_0^*(M) = H^*(\mathfrak{n}, M)^{\mathfrak{a}}$. If $\mathfrak{a}$ acts reductively on a short-exact sequence of s-modules, the exactness of the functor $(\cdot)^{\mathfrak{a}}$ gives a long-exact sequence in *invariant-cohomology* i.e. for $H_0^*(\cdot)$, because $\mathfrak{a}$ acts reductively on $\Lambda^*(\mathfrak{n}^*)$ in any case.

A key point in the computation of invariant cohomology is the following observation.

PROPOSITION 3.8

$H_0^p(\Lambda^q(\mathfrak{n})) \simeq H^{q,p}(G/T:\mathbb{C})$: *in particular, these groups are zero for* $q \neq p$.

Proof. Only the isomorphism requires proof: the fact that the cohomology of G/T is all of type (p, p) is well-known. By definition, $H_0^p(\Lambda^q(\mathfrak{n})) = H^p(\mathfrak{n}, \Lambda^q(\mathfrak{n}))^{\mathfrak{a}}$ and from lemma 3.5 this last space is $H^p(\mathfrak{n}, \Lambda^q(\mathfrak{n}))^{\mathfrak{h}}$. Now theorem II of Bott [3] says

$$H^*(G/T) = H^*(\mathfrak{b}, \mathfrak{h}:\Lambda^*(\mathfrak{n}))$$

and an argument using the Hochschild–Serre spectral sequence, given in Bott [3], Corollary 2 in § 5 (page 223) says

$$H^*(\mathfrak{b}, \mathfrak{h}:\Lambda^*(\mathfrak{n})) = H^*(\mathfrak{n}, \Lambda^*(\mathfrak{n}))^{\mathfrak{h}}.$$

However, we can give a more naive argument for the isomorphism of the proposition. Since G/T is Kahler, the $\bar{\partial}$-cohomology can be computed from the deRham complex: and using the Chevalley–Eilenberg theorem for compact homogeneous spaces

$$H^*(G/T:\mathbb{C}) = H^*(\mathfrak{g}^c, t^c).$$

The chain complex which computes this relative cohomology is $(\Lambda^*(\mathfrak{g}^c/t^c)^{*t}, d)$ with d being the restriction of the differential in the full lie-algebra complex $(\Lambda^*(\mathfrak{g}^c)^*, d)$. The chain complex which computes $H^*(\mathfrak{n}, \Lambda^*(\mathfrak{n}))^{\mathfrak{h}}$ is given by $((\Lambda^*(\mathfrak{n}^*) \otimes \Lambda^*(\mathfrak{n}))^{\mathfrak{h}}, d')$ with d' induced from the differential in the complex which computes the cohomology of $\mathfrak{n}$ with coefficients in the $\mathrm{ad}(\mathfrak{n})$-module $\Lambda^*(\mathfrak{n})$. There is an obvious identification of the chain groups in the two complexes: under this map $d \neq d'$ in general, but their "Laplacians" have the same kernels, as shown by Kostant† [8]. This gives the required isomorphism. //

Remark 3.9. In the following sections we shall need a "topological" description of $H^*(G/T:\mathbb{C})$ which we record below.

† See in particular § 3.1, theorems 4.5 and 5.2.

(a) Let $t^* =$ dual of the lie algebra of T, and W the Weyl group of G. Then $H^*(G/T:\mathbb{R}) \simeq S^*(t^*)/I_W$, where $S^*(t^*) =$ symmetric algebra of t^* over $\mathbb{R}, I_W =$ ideal generated by W-invariants of positive degree. Here a polynomial $f \in S^q(t^*)$ gives a cohomology class in $H^{2q}(G/T:\mathbb{R})$ and upon complexifying, the Dolbeault decomposition is all of type (q, q).

(b) We give an explicit isomorphism $\psi: \mathfrak{h}^* \xrightarrow{\sim} H^2(\mathfrak{g}^c, t^c) = H^2(G/T:\mathbb{C})$: which will be needed later.

$$\psi(\theta) = -\sum_{\alpha>0} \langle \alpha, \theta \rangle \xi_\alpha \wedge \eta_\alpha.$$

Proof that ψ is an isom. Choose a Weyl basis $\{H_j, X_\alpha, Y_\alpha\}$ for $\mathfrak{g}^c$ as above and let $\theta_j \in \mathfrak{h}^*$ be dual to H_j. Then $\psi(\theta_j) \in \Lambda^2(\mathfrak{g}^c)^*$ is precisely $d\theta_j$: being t-basic, $\psi(\theta_j) \in \Lambda^2(\mathfrak{g}^c, t^c)$ and since $\psi(\theta_j)$ is exact in $\Lambda^2(\mathfrak{g}^c)^*$, it is a cocycle in $\Lambda^2(\mathfrak{g}^c, t^c)$. Now $\Lambda^1(\mathfrak{g}^c, t^c) = 0$ since t has no invariants in $\mathfrak{g}^c/t^c$, and hence there are no non-zero coboundaries in $\Lambda(\mathfrak{g}^c, t^c)$. Thus $H^2(\mathfrak{g}^c, t^c) = Z^2(\mathfrak{g}^c, t^c) \subseteq \Lambda^2(\mathfrak{g}^c, t^c)$, and so ψ maps $\mathfrak{h}^*$ to $H^2(\mathfrak{g}^c, t^c)$.

We claim that ψ is $1-1$, onto. If $\Sigma c_j \theta_j \in \ker \psi$, then $d(\Sigma c_j \theta_j) = 0$ which means $\Sigma c_j \theta_j \in \Lambda^1(\mathfrak{g}^c)^*$ is a cocycle: but $H^1(\mathfrak{g}^c) = 0$ by semi-simplicity, and since there are no coboundaries, $\Sigma c_j \theta_j = 0$. At this point we could count dimensions to conclude surjectivity, but here is a more direct argument. If $\omega = \Sigma c_\alpha \xi_\alpha \wedge \eta_\alpha \in \Lambda^2(\mathfrak{g}^c, t^c)$ is a cocycle, then it is a cocycle in $\Lambda^2(\mathfrak{g}^c)^*$ as well, and hence a coboundary since $H^2(\mathfrak{g}^c) = 0$: i.e. $\omega = d\varphi$ for $\varphi \in \Lambda'(\mathfrak{g}^c)$. Since ω is of weight zero (re. t) so is φ, whence $\varphi = \Sigma a_j \theta_j$ for some $a_j \in \mathbb{C}$. //

In 2.7.1 we gave a topological description of $m_2(G)$ which implies that it has real dimension $2k^2$. We shall now check this by computing the Kodaira–Spencer "tangent-space" to $m_2(G)$—namely $H^1(G, \Theta_J)$ where $\Theta_J =$ sheaf of holomorphic tangent vectors to G in the LICS given by J. Again theorem II of Bott [3], gives

$$H^p(G, \Theta_J) = H^p(s, \mathfrak{g}^c/s)$$

and from 3.2 we have $\mathfrak{g}^c/s \simeq \mathfrak{m}^*$ as $\mathrm{ad}(s)$ modules. The short-exact sequence

$$0 \to \mathbb{C}^k \to \mathfrak{m}^* \to \mathfrak{n}^* \to 0$$

gives

$$\to H^0(s, \mathfrak{n}^*) \to H^1(s, \mathbb{C}^k) \to H^1(s, \mathfrak{m}^*) \to H^1(s, \mathfrak{n}^*) \to$$

and using 3.4, 3.5 we see that the end terms are zero.

$$H^0(s, \mathfrak{n}^*) = H^1(s, \mathfrak{n}^*) = 0.$$

Therefore $H^1(s, \mathfrak{m}^*) \simeq H^1(s, \mathbb{C}^k) \simeq H^1(s, \mathbb{C}) \otimes \mathbb{C}^k \simeq \mathfrak{a}^* \otimes \mathbb{C}^k$ which has complex dimension k^2 and real dimension $2k^2$. Similarly $H^2(G, \Theta) = H^2(s, \mathfrak{m}^*) = 0$ so the deformations are unobstructed.

4. Invariant cohomology: The rank 2 case

In this section we shall compute $H_0^*(\Lambda^*(\mathfrak{m})) = H^*(\mathfrak{n}, \Lambda^*(\mathfrak{m}))^{\mathfrak{a}}$ for G a compact simple group of rank 2. By 3.4, this gives a computation of $H_{\bar{\partial}}^{**}(G)$ and along the way

corrects an error of computation in Bott [2]. The calculations can be done in a simple uniform manner (without using the case by case consideration of simple rank 2 groups) and they given a rather vivid demonstration of the dependence of $H_{\bar\partial}(G)$ on moduli. Moreover we can introduce the basic tools of the next section without the notational complexity of the general case.

Recall from 3.2 that we have an extension of ad(s)-modules:

$$0 \to \mathfrak{n} \to \mathfrak{m} \to \mathbb{C} \to 0,$$

where $\mathfrak{n}$ is a maximal nilpotent subalgebra of $\mathfrak{g}^c$. Let $n = \dim_{\mathbb{C}} \mathfrak{n}$. For each $q \geqslant 1$ we have exact sequences of ad(s)-modules,

$$0 \to \Lambda^q(\mathfrak{n}) \to \Lambda^q(\mathfrak{m}) \to \mathbb{C} \otimes \Lambda^{q-1}(\mathfrak{n}) \to 0 \tag{4.1}$$

(see 4.1 or Hirzebruch [5] pg. 55) and from this a long-exact sequence for the functor $H_0^*(\cdot)$. Using 3.8 we have

PROPOSITION 4.2

(a) $H_0^p(\Lambda^q(\mathfrak{m})) = 0$ *unless* $p = q-1$ *or* q

(b) *in the remaining cases there is an exact sequence*

$$0 \to H_0^{q-1}(\Lambda^q(\mathfrak{m})) \to \mathbb{C} \otimes H^{q-1,q-1} \xrightarrow{\delta_{q-1}} H^{q,q} \to H_0^q(\Lambda^q(\mathfrak{m})) \to 0. \qquad //$$

(*Here* $H^{p,p} = H^{p,p}(G/T:\mathbb{C}) = H^{2p}(G/T:\mathbb{C})$).

Thus our task is simply to make δ_{q-1} explicit and then to calculate its kernel and cokernel. Recall first that the de Rham cohomology algebra $H^*(G/T:\mathbb{C})$ is described as

$$H^*(G/T:\mathbb{C}) = S^*(\mathfrak{h}^*)/I_W. \qquad \text{(see 3.8)}$$

It is known in general that the ideal I_W has $l(=\text{rank } G)$ independent generators: but in the rank 2 case W must be a dihedral group and then the algebraic independence of the generators is even more classical. Indeed since W acts absolutely irreducible on t, the first polynomial invariant of W on $S^*(\mathfrak{h}^*)$ is a non-degenerate quadratic form. We may choose co-ordinates $z, \bar z$ on $\mathfrak{h}$ so that this form is given by $z \cdot \bar z$: and then (multiplying z by some $\exp(i\theta)$ if necessary) the second generating invariant can

be chosen to be $z^n \pm \bar z^n$, where $n = \dim_{\mathbb{C}} \mathfrak{n}$[†]. Thus if G is compact, simple of rank 2

$$H^*(G/T:\mathbb{C}) = \mathbb{C}[z, \bar z]/(z\bar z, z^n \pm \bar z^n) \quad z, \bar z \in H^{1,1}.$$

It follows that for p in the range $1 \leqslant p \leqslant n-1$, $H^{p,p}$ is two-dimensional, generated by z^p and $\bar z^p$ whereas $H^{n,n}$ is generated by z^n(or $\bar z^n$).

Now recall that $H^{1,1} \simeq \mathfrak{h}^*$ by 3.9, and the explicit isomorphism ψ given there. Then

[†] Group-theoretically, the degree of the second generator is given in terms of W: one identifies it with n either by lie algebraic facts or from the Poincaré-series of G/T.

PROPOSITION 4.3

$\delta_0:\mathbb{C}\to H^{1,1}\simeq\mathfrak{h}^*$ identifies $\mathbb{C}$ with $\bar{\mathfrak{a}}^*\subseteq\mathfrak{h}^*$ and $\delta_q:\mathbb{C}\otimes H^{q,q}\to H^{q+1,q+1}$ is given by

$$\delta_q(1\otimes\varphi)=\delta_0(1)\varphi.$$

Proof. A more general version is given in 5.5: in any case, the calculation amounts to applying the usual definition of the coboundary map associated to a short-exact sequence of chain complexes. //

Let $\delta_0(1)=w=\alpha z+\beta\bar{z}\in H^{1,1}$, where $\alpha,\beta\in\mathbb{C}$. The condition that $\mathfrak{a},\bar{\mathfrak{a}}$ span $\mathfrak{h}$ says that $w,\bar{w}$ are linearly independent over $\mathbb{C}$, so $w\wedge\bar{w}=(|\alpha|^2-|\beta|^2)z\wedge\bar{z}\neq 0$. Note that $\mathfrak{a}\subseteq\mathfrak{h}$ represents an isotropic line iff $\alpha\beta=0$.

PROPOSITION 4.4

If $\mathfrak{a}\subseteq\mathfrak{h}$ is anisotropic,

$$\delta_q:\mathbb{C}\otimes H^{q,q}\to H^{q+1,q+1}$$

is surjective for $1\leqslant q\leqslant n-1$. If $\mathfrak{a}$ is isotropic then δ_q has a 1-dimensional kernel and hence a 1-dimensional image (in the same range, $1\leqslant q\leqslant n-1$).

Proof. δ_q is multiplication by $w=\alpha z+\beta\bar{z}$. Hence in the specified range, $\delta_q(z^q)=\alpha z^{q+1}$, $\delta_q(\bar{z}^q)=\beta\bar{z}^{q+1}$ since all mixed terms are zero (because $z\bar{z}$ is in the ideal of relations). It remains to note that $\alpha\beta\neq 0$ iff w is anisotropic. //

We will prove something stronger in a moment. Combining 4.2 with 4.4 we see that for $1\leqslant q\leqslant n-1$, $H_0^{q-1}(\Lambda^q(\mathscr{m}))$ and $H_0^q(\Lambda^q(\mathscr{m}))$ are both zero if $\mathfrak{a}$ is anisotropic, 1-dimensional if $\mathfrak{a}$ is isotropic; this shows the strong dependence on moduli.

The proper way to formulate and prove 4.4 is to notice that 4.2(b) gives a bigraded cochain complex

$$C^{p,q}=\Lambda^{q-p}(\mathbb{C})\otimes H^{p,p},\quad \delta^{p,q}=\delta_{q-p}$$

whose cohomology is $H_0^*(\Lambda^*(\mathscr{m}))$. Putting $t\in\Lambda^1(\mathbb{C})$ to be the generator "1" of 4.3, $\delta(t)=w=\alpha z+\beta\bar{z}$ and hence $H^{*,*}(C,\delta)$ is computed from

(i) $H^*(G/T:\mathbb{C})/(w)$ giving $\oplus H_0^p(\Lambda^p(\mathscr{m}))$,

(ii) Closed forms $t\otimes\zeta,\zeta\in H^{p-1,p-1}$, giving $H_0^{p-1}(\Lambda^p(\mathscr{m}))$,

and now 4.4 can be reformulated as

PROPOSITION 4.4′

If $\mathfrak{a}$ is isotropic (so we can take $\alpha=1$, $\beta=0$)

$$\oplus H_0^p(\Lambda^p(\mathscr{m}))=\mathbb{C}[\bar{z}]/(\bar{z}^n),\quad \oplus H_0^{p-1}(\Lambda^p(\mathscr{m}))=\{t\otimes\bar{z}^{p-1}\}$$

if $\mathfrak{a}$ is anisotropic

$$\oplus H_0^p(\Lambda^p(\mathscr{m}))=\Lambda^*(z),\quad \oplus H_0^{p-1}(\Lambda^p(\mathscr{m}))=0\quad\text{for}\quad p\neq n$$

and $H_0^{n-1}(\Lambda^n(\mathscr{m}))=\{t\otimes z^n\}$.

Proof. The calculation of the quotient ring $H^*(G/T:\mathbb{C})/(w)$ in the two cases is routine algebra. If $\delta(t\otimes az^p + b\bar{z}^p) = 0$ then we must have $p = n$ or if $p \leqslant n-1$ $\alpha a = 0$ and $\beta b = 0$. In the isotropic case, this says (for $p \leqslant n-1$) that $a = 0$, and in the anisotropic case, $a = b = 0$. //

From this last result the algebra structure of the invariant ring $H_0^*(\Lambda^*(\mathscr{m}))$ can be read off at once. The only point to observe, is that in the isotropic case, if we put

$$y_{1,1} = [\bar{z}] \text{ the cohomology class of } \bar{z} \text{ in } H_0^1(\Lambda^1(\mathscr{m}))$$

$$v_{2,1} = [t\otimes\bar{z}] \text{ the cohomology class of } t\otimes\bar{z} \text{ in } H_0^1(\Lambda^2(\mathscr{m})).$$

Then

$$[t\otimes\bar{z}^p] = v_{2,1}\cdot y_{1,1}^{p-1}.$$

Combining these observations with 3.4 we obtain

PROPOSITION 4.5

If G is compact, simple of rank 2, *and* $\mathbb{C}-\dim n+1$

$$H_{\bar{\partial}}(G) = \begin{cases} \Lambda^*(x_{0,1}, y_{1,1}, u_{n,n-1}) & \text{if } s \text{ is anisotropic} \\ \Lambda^*(x_{0,1}, u_{2,1})\otimes\mathbb{C}[y_{1,1}]/(y_{1,1})^n & s \text{ isotropic.} \end{cases}$$ //

Since n must be $\geqslant 3$ we see how different the ring structure for a given G can be according as s is isotropic or not. In particular the structure of $H_{\bar{\partial}}(G)$ varies *within* a component of moduli, as each component contains isotropic and anisotropic structures (see 2.6). Denoting by $h^{p,q}(G) = \dim_{\mathbb{C}} H^{p,q}(G)$ we see that for any simple, rank 2 group G, the analogue of the 4th Betti number

$$\sum_{p+q=4} h^{p,q}(G) = \begin{cases} 0 & \text{if } s \text{ is anisotropic,} \\ 1 & \text{if } s \text{ is isotropic.} \end{cases}$$

It is worth remarking here that the case of $G = SU(3)$ is considered by Bott [3] (Example on page 205); but even granting the typographical mislabelling of the $h^{p,q}$ there, he seems to have considered only the isotropic case, (despite a claim to the contrary) because in his example the Hodge numbers do not vary with moduli, and agree with our computations (after proper relabelling of $h^{p,q}$) only in the isotropic case.

It is possible to include the case of $G = \mathrm{Spin}(4)$ in the statement of Proposition 4.5 with a little extra work. The simplicity of G was used only in describing $H^*(G/T:\mathbb{C})$ in terms of the invariants of the Weyl group. For $G = \mathrm{Spin}(4)$, $G/T \simeq S^2 \times S^2$ and so we can compute the maps δ_q for $q = 0, 1, 2$ directly: and we obtain

$$H_{\bar{\partial}}(\mathrm{Spin}(4)) = \Lambda^*(x_{0,1}, y_{1,1}, u_{2,1}) \tag{4.6}$$

irrespective of whether the LICS is isotropic or not: this agrees formally with the result in 4.5 since $n = 2$. Note also that $\mathrm{Spin}(4) \simeq S^3 \times S^3$ can be endowed with an Eackmann–Calabi complex structure: the Dolbeault cohomology of these structures was computed by Borel [2] and agrees with (4.6).

Looking ahead to the Hodge–de Rham spectral sequence for G, we can easily check that for any of the groups considered in this section $\partial(x_{0,1}) = y_{1,1}$ (up to a non-zero scalar). Hence for the *simple* groups of rank 2 there is only one ∂-closed element of total degree 3: it is $\{x_{0,1}\wedge y_{1,1}\}$ having bidegree $(1,2)$ if s is anisotropic, and it is

$\{u_{2,1}\}$ having bidegree (2, 1) if s is isotropic. In either case $E_2 = E_\infty$ since they have the same dimension as complex vector-spaces.

5. Invariant cohomology: The general case

Our first result is that in precise analogy with the rank 2 case, there is a bigraded complex

$$B^{*,*} = \Lambda^*(\bar{\mathfrak{a}}^*) \otimes H^{*,*}(G/T:\mathbb{C}),$$

whose cohomology is the invariant cohomology ring $H_0^*(\Lambda^*(\mathfrak{m}))$. This permits us to calculate E_3 in Borel's tri-graded spectral sequence [2] associated to the holomorphic bundle $T \to G \to G/T$ (see 2.8) and we prove that $E_3 = E_8$ for this spectral sequence.

Finally we make a conjecture (5.12) offering an even simpler complex than $B^{*,*}$ to compute the invariant cohomology, and we verify the conjecture on a special class of LICS on all classical groups of type B_{2k}, C_{2k}, D_{2k}. Consider the short-exact sequence (3.1) of ad(s)-modules:

$$0 \to \mathfrak{n} \to \mathfrak{m} \xrightarrow{\pi} \mathbb{C}^k \to 0.$$

The map π induces a surjective graded-algebra homomorphism $\pi:\Lambda^*(\mathfrak{m}) \to \Lambda^*(\mathbb{C}^k)$ which is also a map of ad(s)-modules. Thus we have a short-exact sequence of ad(s) modules

$$0 \to I \xrightarrow{*} \Lambda^*(\mathfrak{m}) \xrightarrow{\pi} \Lambda^*(\mathbb{C}^k) \to 0,$$

where $I^* = \ker \pi$ is the graded ideal in $\Lambda^*(\mathfrak{m})$ generated by $\Lambda^1(\mathfrak{n})$. Now I^* has a filtration by graded ideals which are also ad(s) modules:

$$I_r^* = \text{ideal in } \Lambda^*(\mathfrak{m}) \text{ generated by } \Lambda^r(\mathfrak{n})\ r \geqslant 0.$$

Thus $I_0^* = \Lambda^*(\mathfrak{m})$ and $I_1^* = I^*$. Denoting by $I_r^q =$ component in degree q of I_r^*, we have

$$I_r^q = 0 \quad \text{for} \quad q < r, \quad I_q^q = \Lambda^q(\mathfrak{n}) \quad \text{and for} \quad q \geqslant r$$

$$0 \to I_{r+1}^q \to I_r^q \to \Lambda^{q-r}(\mathbb{C}^k) \otimes \Lambda^r(\mathfrak{a}) \to 0 \tag{5.1}$$

from which one sees that $I_r^q = \Lambda^q(\mathfrak{m})$ if $q - r \geqslant k + 1$, and that for $k = 1$ (5.1) specializes to (4.1).

From 3.8, the right most term of (5.1) has invariant-cohomology only in degree r: hence

$$H_0^p(I_{r+1}^q) \simeq H_0^p(I_r^q) \quad \text{for} \quad p \neq r, \quad r+1. \tag{5.2}$$

PROPOSITION 5.3

(a) $H_0^p(I_r^q) = 0$ for $p \leqslant r$
(b) $H_0^p(I_r^q) \simeq H_0^p(\Lambda^q(\mathfrak{m}))$ for $p \geqslant r$.

Proof. Fix the indices q and r, with $r \leqslant q$. If $p \leqslant r$, then from (5.2) we see that for any $t \geqslant 0$ $H_0^p(I_{r+t}^q) \simeq H_0^p(I_r^q)$ and so putting $t = q - r$ we have $H_0^p(I_r^q) \simeq H_0^p(I_q^q) = H_0^p(\Lambda^q(\mathfrak{n})) = 0$ since $p \leqslant q$. This proves (a).

Part (b) follows from definition if $r=0$: taking $p>r\geqslant 1$ (5.2) applied with r replaced by $r-t$, $(t\geqslant 1)$ shows $H_0^p(I_r^q)\simeq H_0^p(I_{r-t}^q)$ and $I_0^q=\Lambda^q(\mathfrak{m})$. //

From the proposition one can easily deduce that $H_0^p(\Lambda^q(\mathfrak{m}))=0$ unless $q-k\leqslant p\leqslant q$: however, this will turn up again in a moment. Given 5.3, what we are lacking is a computation of $H_0^r(I_r^q)$. The long-exact sequence for $H_0^*(,)$ associated to (5.1) using 5.3 gives

$$0\to H_0^r(I_r^q)\to\Lambda^{q-r}(\mathbb{C}^k)\otimes H^{r,r}\xrightarrow{\delta^q}H_0^{r+1}(I_{r+1}^q)\to H_0^{r+1}(\Lambda^q(\mathfrak{m}))\to 0, \qquad (5.4)$$

where $H^{r,r}=H^{r,r}(G/T:\mathbb{C})$. Splicing these sequences for two consecutive values of r in tandem, we can identify $H_0^{r+1}(I_{r+1}^q)$ as the kernel of δ_{r+1}^q on $\Lambda^{q-r-1}(\mathbb{C}^k)\otimes H^{r+1,r+1}$ and thus regard $H_0^*(\Lambda^*(\mathfrak{m}))$ as the cohomology of the following "semi-Koszul" complex. Define a bigraded algebra $B^{*,*}$ with a map δ of bi-degree $(1,0)$ by

$$B^{i,j}=\Lambda^{j-i}(\mathbb{C}^k)\otimes H^{i,i}(G/T:\mathbb{C})\quad \delta=\delta_i^j.$$

We will now give a formula for δ_i^j which will prove $\delta^2=0$. Fix a basis $e_1,\ldots,e_k$ for $\mathbb{C}^k$ and lift e_j to $\mathfrak{m}$ to give a basis $\theta_1,\ldots,\theta_k$ of $\bar{\mathfrak{a}}^*=$ annihilator of $\mathfrak{a}\leqslant\mathfrak{h}$. Recall also the isomorphism $\psi:\mathfrak{h}^*\xrightarrow{\sim}H^{1,1}(G/T:\mathbb{C})$ of §3.

PROPOSITION 5.5

For $p=q-r$, $\varphi\in H^{r,r}$

$$\delta_r^q(i_{i_1}\wedge\cdots\wedge e_{i_p}\otimes\varphi)=\sum_{j=1}^{p}(-1)^j e_{i_1}\wedge\cdots\hat{e}_{i_j}\cdots\wedge e_{i_p}\otimes\psi(\theta_{i_j})\varphi.$$

Proof. Consider the projection $I_r^q\to\Lambda^{q-r}(\mathbb{C}^k)\otimes\Lambda^r(\mathfrak{n})$ of 5.1; regarding I_r^q as a submodule of $\Lambda^q(\mathfrak{m})$ we can lift the element $e_{i_1}\wedge\cdots\wedge e_{i_p}\otimes\varphi$ to $\theta_{i_1}\wedge\cdots\wedge\theta_{i_p}\cdot\tilde{\varphi}$ where $\tilde{\varphi}$ is closed† for the differential $d_{\mathfrak{m}}$ of $C^*(\mathfrak{n},\Lambda^q(\mathfrak{m}))^{\alpha}$. Hence it suffices to compute $d_{\mathfrak{m}}:\Lambda^q(\mathfrak{m})\to C^1(\mathfrak{n};\Lambda^q(\mathfrak{m}))$. Since $d_{\mathfrak{m}}$ is a derivation, it suffices to compute $d_{\mathfrak{m}}(\theta_j)$: but this is the differential restricted from the complex for $\mathfrak{g}^c$. Hence

$$d_{\mathfrak{m}}(\theta_j)=\sum_{\alpha}\langle\theta_j,\alpha\rangle\quad \xi_\alpha\wedge\eta_\alpha=\psi(\theta_j)$$

and the rest follows.

PROPOSITION 5.6

$H^{i,j}(B^{*,*},\delta)=H_0^i(\Lambda^j(\mathfrak{m}))$.

Proof. Knowing $\delta^2=0$ from 5.5, this follows from 5.4. //

Note that the E_2-term of the de Rham spectral sequence for the bundle $T\to G\to G/T$ is given by

$$E_2=\Lambda^*(\mathfrak{h}^*)\otimes H^*(G/T:\mathbb{C}),\quad d_2 \text{ on } \Lambda^1(\mathfrak{h}^*) \text{ is } \psi$$

† By formal "harmonic theory".

and this specifies d_2. Hence $(B^{*,*}, \delta)$-as graded by total degree, is a sub complex of (E_2, d_2).

COROLLARY 5.6.1

(a) $H_0^i(\Lambda^j(\mathfrak{m})) = 0$ *unless* $j - k \leqslant i \leqslant j$,
(b) $H_0^0(\Lambda^j(\mathfrak{m})) = 0$ *for* $j \geqslant 1$.

Proof. (a) follows since $B^{i,j}$ is zero outside the same range. For (b) the only cases not covered by (a) are $1 \leqslant j \leqslant k$. Now

$$H_0^0(\Lambda^j(\mathfrak{m})) = \mathrm{Ker}(\Lambda^j(\mathbb{C}^k) \xrightarrow{\delta_0^1} \Lambda^{j-1}(\mathbb{C}^k) \otimes H^{1,1}).$$

From 3.8 $H^{1,1} = \mathfrak{h}^* = \mathfrak{a}^* + \bar{\mathfrak{a}}^*$, and the module $\mathbb{C}^k$ above is $\bar{\mathfrak{a}}^*$. Moreover from 5.5 the map δ_0^1 factors as

$$\Lambda^j(\bar{\mathfrak{a}}^*) \xrightarrow{\delta} \Lambda^{j-1}(\bar{\mathfrak{a}}^*) \otimes \bar{\mathfrak{a}}^* \hookrightarrow \Lambda^{j-1}(\bar{\mathfrak{a}}^*) \otimes \mathfrak{h}^*,$$

where δ is the Koszul differential in the acyclic complex $(\Lambda^*(\bar{\mathfrak{a}}^*) \otimes S^*(\bar{\mathfrak{a}}^*), \delta)$ and hence is injective //

Notice that (b) is just 3.7 and that (a) is essentially forced by 3.4 and Serre duality.

To make further calculations in $H(B^{*,*}, \delta)$ we introduce the following "resolution" of the complex B. From 3.9 (a) we have

$$0 \to I_W^* \to S^*(\mathfrak{h}^*) \to H^*(G/T : \mathbb{C}) \to 0.$$

Tensoring this sequence with $\Lambda^*(\bar{\mathfrak{a}}^*)$ and introducing the "obvious" Koszul type differential (see below) we obtain a short-exact sequence of bigraded differential algebras.

$$0 \to I_W^{*,*} \xrightarrow{\eta} K^{*,*} \xrightarrow{\rho} B^{*,*} \to 0 \tag{5.7}$$

where $K^{i,j} = \Lambda^{j-i}(\bar{\mathfrak{a}}^*) \otimes S^i(\mathfrak{h}^*)$,

$$I_W^{i,j} = \Lambda^{j-i}(\bar{\mathfrak{a}}^*) \otimes I_W^i$$

and the differential d is specified by requiring it to be a derivation of type $(1,0)$ which maps $K^{0,1} B^{0,1}$ to $K^{1,1} = B^{1,1}$ agreeing with δ of 5.5. Note that the bigrading has been chosen to conform to that of $B^{*,*}$ so that $K^{i,j} = I_W^{i,j} = 0$ for $i > j$.

Lemma. $H^{*,*}(K, d) = H^{i,i}(Kd) = S^i(\bar{\mathfrak{a}}^*)$.

Proof. Write $S^*(\mathfrak{h}^*) = S^*(\bar{\mathfrak{a}}^*) \otimes S^*(\mathfrak{a}^*)$. Then $K^{*,*}$ is the tensor product of the acyclic Koszul complex $(\Lambda^*(\bar{\mathfrak{a}}^*) \otimes S^*(\bar{\mathfrak{a}}^*), d)$ with the trivial complex $S^*(\mathfrak{a}^*)$: hence the result follows from the Künneth formula. //

PROPOSITION 5.8

(a) *For* $i < j$, $H^{i-1,j}(B) \simeq H^{i,j}(I_W)$

(b) *There are exact sequences*

$$0 \to H^{j-1,j}(B) \to H^{j,j}(I_W) \xrightarrow{\eta_*} S^j(\mathfrak{a}^*) \xrightarrow{\rho_*} H^{j,j}(B) \to 0.$$

Proof. The maps η_*, ρ_* in 5.7 preserve bidegree, and the coboundary map has degree $(1,0)$: combined with the lemma this completes the proof. //

Notice the analogy between 5.8 and 4.2. As a first consequence of this proposition we have

COROLLARY 5.8.1

$\bigoplus_p H^{p,p}(B) = S^*(\mathfrak{a}^*)/I_W^*(\mathfrak{a})$ *where* $I_W^*(\mathfrak{a})$ *is the ideal in* $S^*(\mathfrak{a}^*)$ *given by* $I_W^* \to S^*(\mathfrak{h}^*) \to S^*(\mathfrak{a}^*)$, *the second map coming from the inclusion of spaces* $\mathfrak{a} \subset \mathfrak{h}$. //

At the risk of being repetitious, we mention that 5.8 gives the "best" proof of 3.7. i.e. $H^{p,0}(G) = 0$ for $p \geqslant 1$. First of all, from 3.4, $H^{p,0}(G) = H_0^0(\Lambda^p(\mathfrak{m})) = H^{0,p}(B)$: and from 5.8 $H^{0,p}(B)$ is a subspace of $H^{1,p}(I_W)$ for $p \geqslant 1$. But the ideal I_W is zero in degree 1 since W acts irreducibly on $\mathfrak{h}$: hence the non-existence of holomorphic p-forms on G is reduced to a "topological" assertion ($H^1(BG:\mathbb{R}) = 0$, for example).

Note that for $p > 1$, $H^{p,p}(G) \neq H_0^p(\Lambda^p(\mathfrak{m}))$ in general.

COROLLARY 5.8.2

If $n = \dim_{\mathbb{C}} G/T$, $H_0^p(\Lambda^p(\mathfrak{m})) = 0$ *for* $p \geqslant n$.

Proof. Since $\oplus H_0^p(\Lambda^p(\mathfrak{m}))$ is a quotient of $H^*(G/T:\mathbb{C})$, the assertion is clear for $p > n$. For $p = n$, $H_0^n(\Lambda^n(\mathfrak{m}))$ is a quotient of $H^{2n}(G/T:\mathbb{C})$, which is contained in the ideal $(\bar{\mathfrak{a}}^*) \subseteq H^*(G/T:\mathbb{C})$ by Poincaré-duality. //

If we extend the DGA $(B^{*,*}, \delta)$ to $(\Lambda^*(\mathfrak{a}^*) \otimes B^{*,*}, \delta)$ by requiring δ to be a derivation which is 0 on the first factor, then from 3.4 we obtain

PROPOSITION 5.9

$H_{\bar{\partial}}(G) \simeq H(\Lambda^*(\mathfrak{a}^*) \otimes B^{*,*}, \delta)$. //

We have been deliberately vague about the Hodge-types in the statement above, because the grading on $B^{*,*}$ was initially chosen to conform to $H_0^*(\Lambda^*(\mathfrak{m}))$: and in this context there is a "shear" in the Hodge-type which is essentially dictated by Serre duality, coupled with 3.4.

The Hodge-types can be recovered as follows. First recall from 3.6 that $\Lambda^r(\mathfrak{a}^*)$ corresponds to $H_{\bar{\partial}}^{0,r}(G)$ and so we assign to it bidegree $(0,r)$. Then $\Lambda^s(\bar{\mathfrak{a}}^*)$ will have bidegree $(s,0)$ and of course $H^{i,i}$ has bidegree (i,i). Hence

$$\Lambda^r(\mathfrak{a}^*) \otimes B^{i,j} \text{ has bidegree } (j-i,r) + (i,i) = (j, r+i).$$

Now consider Borel's spectral sequence for the holomorphic bundle $T \to G \to {}^{G}/T$. From [2], 7.1 we have

$$ {}^{p,q}E_2^{s,t} \simeq \sum_i H^{i,s-i}(G/T) \otimes H^{p-i,q-s+i}(T). $$

The differential d_2 is a derivation which is determined by its values on ${}^{p,q}E_2^{0,t}$-i.e. $H^{*,*}(T)$- and in this case by its values on ${}^{1,0}E_2^{0,1} = H^{1,0}(T)$ and ${}^{0,1}E_2^{0,1} = H^{0,1}(T)$ since the cohomology of the fibre is exterior on the sum of these two spaces.

PROPOSITION 5.10

$(E_2, d_2) \simeq (\Lambda^*(\mathfrak{a}^*) \otimes B^{*,*}, \delta)$ with the bigrading in the second conplex given as above.

Proof. From the known description of δ, and the remarks above, it is enough to show that

$$ d_2 : {}^{0,1}E_2^{0,1} \to {}^{0,2}E_2^{2,0} = H^{0,2}(G/T:\mathbb{C}) = 0 $$

and

$$ d_2 : {}^{1,0}E_2^{0,1} \to {}^{1,1}E_2^{2,0} \simeq H^{1,1}(G/T:\mathbb{C}) = \mathfrak{h}^* $$

is injective onto $\bar{\mathfrak{a}}^* \subseteq \mathfrak{h}^*$. d_2 on $H^{1,0}(T) = {}^{1,0}E_2^{0,1}$ must be injective: else we would have global $(1,0)$ forms on G, since the kernel of d_2 on this group would persist to ∞. Similarly, since

$$ {}^{1,1}E_3^{2,0} = {}^{1,1}E_\infty^{2,0} $$

and we know $H^{1,1}(G) = H_0^1(\Lambda^1(\mathscr{m})) = \mathfrak{a}^*$, we see that $d_2(H^{1,0}(T))$ is as claimed. //

PROPOSITION 5.11

Borel's spectral sequence for $T \to G \to G/T$ collapses at E_3, and as bigraded algebras,

$$ H_{\bar{\partial}}(G) \simeq E_3 = E_\omega. $$

Proof. The first isomorphism is a consequence of 5.8 and 5.9. Hence $\dim_{\mathbb{C}} E_3 = \dim_{\mathbb{C}} H_{\bar{\partial}}(G)$; and by convergence, $\dim_{\mathbb{C}} E_\infty = \dim_{\mathbb{C}} H_{\bar{\partial}}(G)$. Thus perforce, $d_r = 0$ for $r \geqslant 3$. //

For our remarks on Borel's spectral sequence, the complex B^{**}, or $\Lambda^*(\mathfrak{a}^*) \otimes B^{*,*}$ was quite effective, as it was in the rank 2 case (§4). However, for larger ranks explicit computations of $H^{*,*}(B, \delta)$ become quite laborious; we present here a conjectural complex to compute the invariant cohomology, which will be applied in the next section to work out $H_{\bar{\partial}}(SO(9))$ with a particular LICS.

Fix a basis $f_i, \ldots, f_l$ of homogeneous generators for the polynomial subring $A^* = S^*(\mathfrak{h}^*)^W$ of invariants of the Weyl group in the full polynomial algebra $S^* = S^*(\mathfrak{h}^*)$, Assign the elements of $\mathfrak{h}^* = S^1$ bidegree $(1,1)$ so that bi-deg $f_i = (m_i, m_i)$. Define the bigraded exterior algebra $\Lambda^*(\varphi_1, \ldots, \varphi_l)$ where

$$ \text{bi-deg}\, \varphi_i = (m_i, m_{i-1}) $$

and let $\mathfrak{a} \subseteq \mathfrak{h}$ be a complex subspace of half the dimension corresponding to an LICS on G.

Conjecture 5.12. The invariant cohomology of G is the cohomology of the complex

$$\Lambda^*(\varphi_1,\ldots,\varphi_l)\otimes S^*(\mathfrak{a}^*) \quad \text{with} \quad d\varphi_j = f_j, \quad d(\mathfrak{a}^*) = 0.$$

Equivalently with a switch in the bidgree of φ_j, $H_{\bar\partial}(G) = \Lambda^*(\mathfrak{a}^*)\otimes\Lambda^*(\varphi_1,\ldots,\varphi_l)\otimes S^*(\mathfrak{a}^*)$ with the differential extended from the complex above by setting $d(\Lambda^1(\mathfrak{a}^*)) = 0$.

The computations of § 4 show that 5.12 is true-for groups of rank 2; below we shall give some further examples, but first we describe the motivation for this conjecture. The fibre bundle $T\to G\to G/T$ gives a spectral sequence in deRham theory with

$$E_2 = \Lambda^*(\mathfrak{h}^*)\otimes S^*(\mathfrak{h}^*)/(f_1,\ldots,f_l)$$

$$d_2:\Lambda^1(\mathfrak{h}^*)\xrightarrow{\sim} S^1(\mathfrak{h}^*)$$

abutting to $\operatorname{gr} H^*(G)$. Without benefit of Hopf's theorem, and in a somewhat round about way we can deduce the isomorphism.

$$H^*_{\mathrm{DR}}(G) = \Lambda^*(\varphi_1,\ldots,\varphi_l) \quad \text{(graded by total degree)}$$

as follows.

The ring $H^*(G/T) = S^*(\mathfrak{h}^*)/(f_1,\ldots,f_l)$ is a module over $S^* = S^*(\mathfrak{h}^*)$ in an obvious way and we wish to replace $H^*(G/T)$ by a free S^*-resolution of it. First of all, since A^* is itself a polynomial ring the complex

$$\Lambda^*(\varphi_1,\ldots,\varphi_l)\otimes A^*, \quad \mathrm{d}\varphi_j = f_j$$

is acyclic, and hence graded by total degree gives a free A^*-resolution of $\mathbb{C}$. Now S^* is well-known to be a *free* A^*-module, and from 3.8.

$$H^*(G/T) = S^*\bigotimes_{A^*}\mathbb{C}.$$

Hence the complex

$$\Lambda^*(\varphi_1,\ldots,\varphi_l)\otimes S^*, \quad \mathrm{d}\varphi_j = f_j$$

graded by total degree provides a free resolution of $H^*(G/T)$.

Now consider the appropriately graded double-complex

$$\Lambda^*(\mathfrak{h}^*)\otimes\Lambda^*(\varphi_1,\ldots,\varphi_l)\otimes S^*$$

with

$$d_0^{\mathrm{I}}(\Lambda^1(\mathfrak{h}^*))\xrightarrow{\sim} S^1(\mathfrak{h}^*), \quad d_0^{\mathrm{I}}(\varphi_j) = 0$$

and

$$d_0^{\mathrm{II}}(\Lambda^*(\mathfrak{h}^*)) = 0, \quad d_0^{\mathrm{II}}(\varphi_j) = f_j \quad \text{and} \quad d_0^{\mathrm{II}}(S^*) = 0.$$

Then the first spectral sequence has

$$I_1 = E_2(T\to G\to G/T)$$

and to compute its abutment I_∞ we calculate II_r.

Clearly $II_1 = \Lambda^*(\varphi_1,\ldots,\varphi_l)$, and all further differentials are zero since II_1 is

concentrated in "base degree" zero. This gives $I_\infty = II_1 = \Lambda^*(\varphi_1, \ldots, \varphi_l)$ which implies the required assertion.

In this deRham setting one knows (from an old result of Leray) that $I_2 = E_3(T \to G \to G/T)$ is in fact E_∞. In our conjectural set-up below, this is precisely what is missing.

To compute the cohomology of $B^{*,*} = \Lambda^*(\bar{\mathfrak{a}}^*) \otimes H^*(G/T)$ we mimic the procedure above, replacing $H^*(G/T)$ by the free S^* resolution given above to obtain

$$\Lambda^*(\bar{\mathfrak{a}}^*) \otimes \Lambda^*(\varphi_1, \ldots, \varphi_l) \otimes S^*$$

with $d_0^{\mathrm{I}}(\Lambda^1(\bar{\mathfrak{a}}^*) \xrightarrow{\sim} \bar{\mathfrak{a}}^* \subseteq S^1$, $d_0^{\mathrm{I}}(\varphi_j) = f_j$ and d_0^{II} as above. Then from the lemma preceding 5.8

$$I_1 = \Lambda^*(\varphi_1, \ldots, \varphi_l) \otimes S^*(\mathfrak{a}^*),$$

which is our conjectural complex, whereas

$$II_1 = B^{*,*} \quad d_1^{\mathrm{II}} = \delta.$$

What is missing is the statement that $I_2 \simeq II_2$.

We now verify the conjecture on a class of LICS for which a certain property is satisfied, and we produce examples to show that LICS with these properties exist on every member of even rank in the classical families. Suppose the subspace $\mathfrak{a} \subseteq \mathfrak{h}$ is so chosen that the basic set $\{f_1, \ldots, f_l\}$ of Weyl invariants in $S^*(\mathfrak{h}^*)$ breaks up into two disjoint subsets $\{f_{i_p}, \ldots, f_{i_m}\} \cup \{g_1, \ldots, g_n\}$ such that

(a) $f_{i_j}|_{\mathfrak{a}} = 0 \quad l \leqslant j \leqslant m$

(b) $\{g_1|_{\mathfrak{a}}, \ldots, g_n|_{\mathfrak{a}}\}$ forms a regular sequence in $S^*(\mathfrak{a}^*)$.

PROPOSITION 5.13

Under the conditions (a) *and* (b) *the conjecture 5.12 is true.*

Proof. Define the complex $X_0 = \Lambda^*(\bar{\mathfrak{a}}^*) \otimes S^*(\mathfrak{h}^*)$ with the bigrading above. Then by the lemma preceding 5.8 we have

$$H^*(X_0) = S^*(\mathfrak{a}^*).$$

Now $\{f_{i_p}, \ldots, f_{i_m}\}$ are algebraically independent elements of $S^*(\mathfrak{h}^*)$ and hence they form a regular sequence: moreover the hypothesis (a) implies that they lie in the ideal $(\bar{\mathfrak{a}}^*)$ of $S^*(\mathfrak{h}^*)$ and hence are exact elements in the complex X_0. Hence by a well-known argument (repeated use of theorem 3 of Tate [9], for example) if we put

$$X_1 = \Lambda^*(\bar{\mathfrak{a}}^*) \otimes S^*(\mathfrak{h}^*)/(f_n, \ldots, f_{i_m}).$$

Then

$$H^*(X_1) = \Lambda^*(\varphi_1, \ldots, \varphi_m) \otimes S^*(\mathfrak{a}^*),$$

where the φ_j have the correct bidegree.

The set $\{g_1, \ldots, g_n\}$ still defines a regular sequence in (the second factor of) X_1 because the full set $\{f_1, \ldots, f_l\}$ is algebraically independent: and each g_j is closed in

X_1, but by (b) not exact. Define

$$X_{j+1} = X_j/(g_j) = X_1/(g_1, \ldots, g_j).$$

By induction again on the same result of Tate [9] or rather on its proof one obtains

$$H^*(X_{n+1}) = \Lambda^*(\varphi_1, \ldots, \varphi_m) \otimes S^*(\mathfrak{a}^*)/(g_1, \ldots, g_n)$$

and of course $X_{n+1} = B^{*,*}$.

The conjectural complex under the above hypothesis decomposes as a tensor product.

$$\Lambda^*(\varphi_{i_j}, \ldots, \varphi_{i_m}) \otimes \Lambda^*(\psi_1, \ldots, \psi_n) \otimes S^*(\mathfrak{a}^*)$$

with d being zero on the first factor and $d\psi_j = g_j|_{\mathfrak{a}}$. By the Kunneth formula and theorem 4 of Tate [9] this complex has the same cohomology as X_{n+1} above. //

The added complexity of the general case is that the complement $\{g_1, \ldots, g_n\}$ of the basic invariants restricting to zero on $\mathfrak{a}$ will itself decompose into two subsets: $\{g_1, \ldots, g_p\} \cup \{h_1, \ldots, h_q\}$ $(p+q=n)$ where $\{g_1, \ldots, g_p\}$ is maximal with respect to the property that $\{g_j|_{\mathfrak{a}} | l \leqslant j \leqslant p\}$ is a regular sequence in $S^*(\mathfrak{a}^*)$. Hence multiplication by h_r on the complex

$$Y = \Lambda^*(\bar{\mathfrak{a}}^*) \otimes S^*(\mathfrak{h}^*)/(f_1, \ldots, f_m, g_1, \ldots, g_p)$$

will be injective, but the induced map on

$$H^*(Y) = \Lambda^*(\varphi_1, \ldots, \varphi_m) \otimes S^*(\mathfrak{a}^*)/(g_1, \ldots, g_p)$$

will have a non-trivial kernel and cokernel. The bigrading and the algebraic structure of these subspaces do not seem easy to describe in any general and useful way. In the counter example with $SO(9)$ in § 6, there is only one element of type "h" and the computations can be done by hand.

We now produce examples of subspaces $\mathfrak{a} \subseteq \mathfrak{h}$ on the even rank groups in the classical families of type B, C, D which satisfy the conditions of 5.13. First consider $SO(4k+1)$. Fix an orthonormal basis $e_1, \ldots, e_{2k}$ for (-1) Killing form on a maximal toral subalgebra t in the real lie algebra $so(4k+1)$: and let $x_1, \ldots, x_{2k} \varepsilon_i^*$ be the dual co-ordinates. Then a basis for the Weyl invariant polynomials is given by

$$f_p = \sum_{j=1}^{2k} x_j^{2p} \quad l \leqslant p \leqslant k.$$

Let $\mathfrak{a}$ be the subspace of $\mathfrak{h} = t \otimes \mathbb{C}$ generated by

$$e_j + \sqrt{-1}\, e_{j+k} \quad 1 \leqslant j \leqslant k.$$

This subspace is totally isotropic for f_1, and hence $\mathfrak{a} \cap t = \{0\}$ which shows that $\mathfrak{a}$ defines an LICS. If $u_1, \ldots, u_k$ are co-ordinates on $\mathfrak{a}$ dual to the given basis one checks that

$$f_{2p-1}|_{\mathfrak{a}} = 0 \quad \text{essentially because} \quad (\sqrt{-1})^{4p-2} = -1$$

and that f_{2p} is up to scalars

$$\sum_{j=1}^{k} u_j^{2p}, \quad l \leqslant p \leqslant k.$$

Thus we have an example on $so(4k+1)$ satisfying 5.13; evidently one can produce others by permuting the e_j, $1 \leqslant j \leqslant k$ holding e_{j+n} fixed.

By the well-known isomorphism of $(\mathfrak{h}, W)$ for $SO(4k+1)$ and $S_p(2k)$ (isomorphisms of modules over the Weyl groups) essentially the same examples work in the symplectic case. The case of $SO(4k)$ can also be adapted to the examples above, replacing

$$f_{2k} \text{ by } g = \prod_{1}^{2k} x_j \text{ which restricts to } \prod_{1}^{2k} u_j^2 \text{ on } \mathfrak{a}.$$

Finally we mention that the results of §4 show at once that 5.12 is true for semi-simple groups of rank 2 with arbitrary LICS.

The referee has pointed out a paper of Griffithis [7], saying that his computations in Prop. 5.1 and its corollary on pg. 139 of [7] applied to an LICS on G are at variance with the results above. However, this is not so—a slightly imprecise notation in the Corollary has led to this misconstrual.

The "fundamental fibering" of [7] applied to the case $X = G$ is precisely $T \to G \longrightarrow G/T$ (see 2.8(b)): and the corollary says

$$H^*(G, \pi^*\Omega^*(G/T)) \simeq H^*(T, \mathcal{O}_T) \otimes H^*(G/T, {}^*G/T). \tag{5.14}$$

The right-hand side could not possibly be isomorphic to $H^*(G, \Omega^* G)$ for simple reasons of type (= bidegree) arising from Serre duality: and the discussion surrounding the Corollary makes it clear that 5.14 as stated here is what Griffiths really intended.

Note that the right-hand side of 5.14 is essentially our algebra $B^{*,*}$: and its differential δ can be recovered from the map induced on the left-hand side by $\partial: \Omega^*(G/T) \to \Omega^*(G/T)$.

6. The Hodge–deRham (HdR) spectral sequence

On a compact, complex manifold M, the algebra of global, C^∞ complex-valued differential forms $\mathcal{A}(M)$ admits a bigrading given by the decomposition into Hodge types, and a corresponding decomposition of the deRham differential $d = \partial + \bar{\partial}$. Hence $(\mathcal{A}^{*,*}(M), \partial, \bar{\partial})$ is a double-complex whose associated first spectral sequence is

$$E_1^{p,q} = H^q(M, \Omega_M^p) \Rightarrow H_{\mathrm{DR}}^{p+q}(M), \quad d_1 = \partial.$$

(The second spectral sequence being the complex conjugate). If M is Kähler, then $E_1 = E_\infty$ for any number of equivalent reasons. There is a folk-lore conjecture that for *any* compact, complex M, $d_r = 0$ for $r \geqslant 2$: i.e. $E_2 = E_\infty$. We shall show that for a particular (finite) set of LICS on $SO(9)$ this conjecture is false: at least one $d_r \neq 0$ for $r \geqslant 2$.

We will exhibit a particular LICS on $SO(9)$ which has the property that in the HdR

spectral sequence, the total dimension of the E_2-term is

$$\sum_{p,q} \dim_{\mathbb{C}} E_2^{p,q} = 26.$$

On the other hand, the total dimension of E_∞ is 16 since $H^*_{\mathrm{DR}}(SO(9))$ is an exterior algebra on four generators: thus $d_r \neq 0$ for at least one $r \geqslant 2$. Then we will verify the conjecture 5.12 for this LICS, showing that the truth of 5.12 does not force the HdR sequence to collapse at E_2. Finally we will make some remarks on the special features of this LICS as compared with others on $SO(9)$.

Continuing the notation introduced after 5.13 for $so(4k+1)$ we fix the LICS on $SO(9)$ given by the complex 2-plane $\mathfrak{a}_0 \subseteq \mathfrak{h}$ spanned by $e_1 + e_2 + i\sqrt{2}e_3, e_1 - e_2 + i\sqrt{2}e_4$: this is a permissible choice since $\mathfrak{a}_0$ is totally isotropic for $f_1 = \sum_1^4 x_j^2$ and hence free of real points. Let u_1, u_2 be co-ordinates on $\mathfrak{a}_0$ dual to the given basis. Then for the given LICS, the invariant cohomology is the cohomology of the complex.

$$B^{*,*} = \Lambda^*(t_1, t_2) \otimes \mathbb{C}[x_1, \ldots, x_4]/(f_1, \ldots, f_4)$$

bigraded as in 5.6, with $\delta(t_j) = \bar{u}_j$, $l \leqslant j \leqslant 2$. One checks directly that

$$u_1 = \tfrac{1}{4}(x_1 + x_2 - i\sqrt{2}x_3), \quad u_2 = \tfrac{1}{4}(x_1 - x_2 - i\sqrt{2}x_4).$$

Notice that any translate of $\mathfrak{a}_0 \subseteq \mathfrak{h}$ by an element of the Weyl group will give an isomorphic Dolbeault ring, and an isomorphic E_2-term in the HdR sequence. Thus the computations which follow give a finite set of counter-examples to the HdR conjecture.

In what follows we will need to know $f_j|_{\mathfrak{a}_0}$, $1 \leqslant j \leqslant 4$. Define

$$f(u_1, u_2) = u_1^2 + u_2^2, \quad g(u_1, u_2) = u_1^2 u_2^2$$

in the ring $\mathbb{C}[u_1, u_2]$: and for any two polynomials p_1, p_2 of this ring, denote $p_1 = \lambda p_2 (\lambda \in \mathbb{C}^*)$ by $p_1 \sim p_2$. Then we find that $f_1|_{\mathfrak{a}_0} = 0$ (because $\mathfrak{a}_0$ is totally isotropic) and

$$\begin{aligned} &f_2|_{\mathfrak{a}_0} \sim f^2 \\ &f_3|_{\mathfrak{a}_0} \sim f^3 - 8f_g, \\ &f_4|_{\mathfrak{a}_0} \sim g^2 + \lambda_1 f^4 + \lambda_2 f^2 \cdot g \quad \text{for } \lambda_1, \lambda_2 \in \mathbb{C}. \end{aligned} \tag{6.1.1}$$

Introduce the bigraded complex

$$Y^{*,*} = \Lambda^*(t_1, t_2) \otimes \mathbb{C}[x_1, \ldots, x_4]/(f_1, f_2, f_4)$$

with bigrading and δ as above, Since $[f_1, f_2, f_3, f_4\}$ are algebraically independent, they form a regular sequence and

$$0 \to Y^{*,*} \xrightarrow{f_3} Y^{*,*} \to B^{*,*} \to 0 \tag{6.1.2}$$

is a short-exact sequence of bigraded complexes.

PROPOSITION 6.1

$H^{*,*}(Y,\delta) = \Lambda^*(\theta)\otimes\mathbb{C}[u_1,u_2]/(f^2,g^2)$ *where* $\text{bideg}\,\theta=(1,2)$ *and* $\text{bideg}(u_i)=(1,1)$.

Proof. The argument is similar to 5.13 and follows by repeated application of theorem 3 of Tate [9]. First, the cohomology of $Y_0=\Lambda^*(t_1,t_2)\otimes\mathbb{C}[x_1,\ldots,x_4]$ with $\delta(t_j)=\bar{u}_j$ is $\mathbb{C}[u_1,u_2]$. Since $f_1\in Y_0^{2,2}$ is not a zero divisor and is exact (for example $\delta(t_1\otimes u_1+t_2\otimes u_2)=\lambda f$ for suitable $\lambda\in\mathbb{C}^*$) we have $H^{*,*}(Y_0/(f_1),\delta)=\Lambda^*(\theta)\otimes\mathbb{C}[u_1,u_2]$, θ being the cohomology class of $t_1\otimes u_1+t_2\otimes u_2$.

Next, $\{f_2,f_4\}$ are regular in $Y_0/(f_1)$ and closed but not exact, and $f_2\sim f^2, f_4\sim g^2$ are regular in $\mathbb{C}[u_1,u_2]$. Hence the result follows. //

The difference from 5.13 now appears in the treatment of f_3. It is easy to check that the map in cohomology induced by multiplication by f_3 on Y is essentially multiplication by fg because $f_3\sim 8fg$ and $f^3=0$ in $H^{*,*}(Y)$. Hence we need to understand the annihilator of fg in the ring $\mathbb{C}[u_1,u_2]/(f^2,g^2)$.

PROPOSITION 6.2

The annihilator of fg *is the ideal* (fg) *and the Poincaré polynomial of this ideal is*

$$t^2(1+t+\cdots+t^3)(1+t+\cdots+t^5).$$

Proof. Define $v_i=u_i^2$: then $\mathbb{C}[u_1,u_2]$ is a free module ever $\mathbb{C}[v_1,v_2]$ with basis $\{1,u_1,u_2,u_1u_2\}$ and $\mathbb{C}[v_1,v_2]$ is a free module over $\mathbb{C}[f,g]$ with basis $1,v_1$. Since f^2,g^2 lie in this last ring, it follows that $\mathbb{C}[u_1,u_2]/(f^2,g^2)$ is free over $\mathbb{C}[v_1,v_2]/(f^2,g^2)$ which is free over $\mathbb{C}[f,g]/(f^2,g^2)$ on the corresponding basis. In this last ring the annihilator of fg is clearly generated by f and g: and in view of the freeness, the same is true in $\mathbb{C}[u_1,u_2]/(f^2,g^2)$. To calculate the Poincaré polynomial we use the exact sequence

$$0\to(f,g)\to\mathbb{C}[u_1,u_2]/(f^2,g^2)\to\mathbb{C}[u_1,u_2]/(f,g)\to 0.$$

Using the algebraic independence of f,g (and hence also of f^2,g^2) the Poincaré polynomials of the middle and right-hand terms are

$$\frac{(1-t^4)(1-t^8)}{(1-t)^2} \quad\text{and}\quad \frac{(1-t^2)(1-t^4)}{(1-t)^2},$$

where the subspace $\mathbb{C}u_1+\mathbb{C}u_2$ is assigned degree 1. Hence the polynomial of the kernel is the difference of the two.

Let $A^{*,*}=\Lambda^*(\theta)\otimes(f,g)$ be the annihilator of fg in $H^{*,*}(Y,\delta)$: thus $\oplus A^{j,j}=(f,g)$ and $\oplus A^{j+1,j+2}=\theta\otimes(f,g)$. With this understood, the long-exact sequence in cohomology of 6.1.1, gives

$$0\to\Lambda^{j-i}(\bar{\mathfrak{a}}_0^*)\otimes\mathbb{C}[u_1,u_2]^j/(f^2,fg,g^2)\to H^{i,j}(B)\to A^{i-5,j-6}\to 0.$$

From considerations of bidegree we find

(a) $\oplus H^{j,j}(B)=\mathbb{C}[u_1,u_2]/(f^2,fg,g^2)$

(b) $0\to\theta\otimes H^{j,j}(B)\to H^{j+1,j+2}(B)\to A^{j-4,j-4}\to 0$ (6.2.1)

(c) $H^{j,j+2}(B)\simeq A^{j-5,j-4}$

and $H^{i,j}(B) = 0$ for other pairs, in agreement with the results of 5.6.1 and 5.8.1. To analyse (b) more carefully we will need the fact that $H^{j,j}(B) = 0$ for $j \geqslant 9$. In fact for later purposes we put $R^j = H^{j,j}(B)$ so that $R^* = {}^{\mathbb{C}[u_1,u_2]}/(f^2, fg, g^2)$ is graded by setting $\deg u_i = 1$, and we compute its Poincaré polynomial.

PROPOSITION 6.3

The Poincaré polynomial of R^* is

$$(1-t^2)(1+t^2+t^4)(1+t)^2.$$

Proof. This is similar to the computation of the Poincaré polynomial in 6.2. R^* is free over $\mathbb{C}[v_1, v_2]/(f^2, fg, g^2)$ on $1, u_1, u_2, u_1u_2$ and this latter ring is free over $\mathbb{C}[f, g]/(f^2, fg, g^2)$ or with basis $\{1, v_1\}$. Since the last ring clearly has Poincaré polynomial $(1 + t^2 + t^4)$ the rest follows. //

Thus we find

$$\text{(b')}\quad \theta \otimes R^j \simeq H^{j+1,j+2}(B) \quad 0 \leqslant j \leqslant 5,$$

$$\text{(b'')}\quad 0 \to \theta \otimes R^j \to H^{j+1,j+2}(B) \to A^{j-4,j-4} \to 0 \quad 6 \leqslant j \leqslant 8,$$

$$\text{(b''')}\quad H^{j+1,j+2}(B) \simeq A^{j-4,j-4} \quad 9 \leqslant j \leqslant 14$$

and $H^{k-1,k}(B) = 0$ in the cases not covered above. Now fix elements $\psi_1 \in H^{7,8}(B)$ and $\psi_2 \in H^{9,10}(B)$ which map under (b'') to $f \in A^{2,2}$ and $g \in A^{4,4}$ respectively.

PROPOSITION 6.4

As a graded module over R^*,

$$\oplus H^{k-1,k}(B) \cong \theta \otimes R^* \oplus \psi_1 \otimes {}^{R^*}/(f, g) \oplus \psi_2 \otimes {}^{R^*}/(g).$$

Proof. Pick liftings of $h_1, h_2 \in R^*$ to the ring $\mathbb{C}[u_1, u_2]/(f^2, g^2)$ and define $R^*H^{7,8} + R^*H^{9,10} \to \oplus A^{k,k}$ by $h_1\psi_1 + h_2\psi_2 \to \tilde{h}_1 f + \tilde{h}_2 g$.

This map is well-defined because different liftings will differ by a multiple of fg, and it is clearly surjective. From (b) above it follows that $\{\theta, \psi_1, \psi_2\}$ generate $\oplus H^{k-1,k}(B)$ as an R^*-module.

Now $g \cdot \psi_2$ maps to zero in $A^{8,8}$ and hence is zero by (b'''). For the same reason, $g\psi_1 - f\psi_2 = 0$. Similarly, $f\psi_1$ must be of the form $h\theta, h \in R^8$. Hence the surjection

$$R^*\theta + R^*\psi_1 + R^*\psi_2 \to \oplus H^{k-1,k}(B)$$

factors through the quotient by the relations just given. This quotient is easily seen to be the right-hand module of the assertion, and it has complex dimension 48 because $R^*/(g)$ has complex dimension 16 and $R^*/(f, g)$ has complex dimension 8, by analogues of 6.3. On the other hand $\oplus H^{k-1,k}(B)$ has complex dimension 48 by 6.2, 6.3 and the exact sequence 6.2.1(b) above. //

PROPOSITION 6.5

As a graded module over R^*

$$\oplus H^{k-2,k}(B) \simeq \theta_1 \wedge \psi_1 \otimes R^*/(f, g) + \theta \wedge \psi_2 \otimes R^*/(g).$$

Proof. In analogy with 6.4 the map

$$R^*(\theta \wedge \psi_1) + R^*(\theta \wedge \psi_2) \to \oplus A^{k+1,k}$$

defined by

$$h_1(\theta \wedge \psi_1) + h_2(\theta \wedge \psi_2) \to \tilde{h}_1 f\theta + \tilde{h}_2 g\theta$$

is surjective by the isomorphism 6.2.1(c) and the relations already given in 6.4 show that this map factors as required. A dimension count completes the proof. //

We summarize the isomorphisms of 6.2.1(a), 6.4, 6.5 in the isomorphism below.

$$H^{*,*}(B) \simeq \Lambda^*(\theta) \otimes (R^* \oplus \psi_1 \otimes R^*/(f,g) \oplus \psi_2 \otimes R^*/(g)). \qquad 6.5.1$$

Notice that we have not specified the ring structure completely, the indeterminacy being $f\psi_1 \in (\theta), \psi_1 \wedge \psi_2 \in (\theta)$.

From 3.4 we obtain

$$H_{\bar{\partial}}(SO(9)) = \Lambda^*(\mathfrak{a}_0^*) \otimes \bar{H}^{*,*}(B) \qquad 6.5.2$$

where the bar "——" is simply a formal device to indicate the switch of indices so that as elements of $H_{\bar{\partial}}(SO(9))\theta, \psi_1, \wedge \psi_2$ have bidegrees $(2,1)$, $(8,7)$ and $(10,9)$ respectively.

Theorem 6.5. *For the LICS on SO(9) corresponding to* $\mathfrak{a}_0$, *the* E_2*-term of the Hodge de Rham spectral sequence has dimension 26.*

Proof. Quite generally for any semi-simple compact group G with an arbitrary LICS we must have an isomorphism

$$d_1 : E_1^{0,1} \simeq \Lambda^1(\mathfrak{a}^*) \xrightarrow{\sim} E_1^{1,1} = S^1(\mathfrak{a}^*),$$

where the identification of the domain and target comes from 3.4, and 5.8.1. Indeed, $E_2^{0,1} = \mathrm{Ker}(d_1 : E_1^{0,1} \to E_1^{1,1})$ persists to E_∞ because $E_1^{2,0} = 0$ by 3.6 and hence $E_2^{2,0} = 0$: thus we have

$$\dim E_2^{0,1} = \dim(E_\infty^{0,1}) \leqslant \dim H^1_{\mathrm{DR}}(G:\mathbb{C}) = 0.$$

In the case at hand then, $d_1 : \Lambda^1(\bar{\mathfrak{a}}_0^*) \xrightarrow{\sim} R^1$ and for trivial reasons of degree $d_1\theta = d_1\psi_1 = d_1\psi_2 = 0$. Thus (E_1, d_1) breaks up as $\Lambda^*(\theta) \otimes (C_1 \oplus C_2 \oplus C_3)$ where

$$C_1 = \Lambda^*(\mathfrak{a}_0^*) \otimes R^*, \quad C_2 = \psi_1 \otimes (\Lambda^*(\mathfrak{a}_0^*) \otimes R^*/(f,g))$$
$$C_3 = \psi_2 \otimes (\Lambda^*(\mathfrak{a}_0^*) \otimes R^*/(g)),$$

with the differential in all three cases specified by the remarks directly above. Hence

$$E_2 \simeq \Lambda^*(\theta) \otimes (H(C_1) \oplus H(C_2) \oplus H(C_3)).$$

Now the cohomology of C_2, C_3 is easy to compute from the now familiar result

of Tate [9] because the ideals defining

$$R^*/(f,g)=\mathbb{C}[u_1,u_2]/(f,g) \quad \text{and} \quad R^*/(g)=\mathbb{C}[u_1,u_2]/(f^2,g)$$

are generated by regular exact pairs. Pick a basis s_1, s_2 for $\Lambda^1(\mathfrak{a}_0^*)$ such that $d_1 s_j = u_j$: and define

$$\xi = s_1 \otimes u_1 + s_2 \otimes u_2, \quad \eta = s_1 \otimes u_1 u_2^2$$

so that $d_1\xi = f$, $d_1\eta = g$. Then

$$H(C_2) = \psi_1 \otimes \Lambda^*([\xi],[\eta]),$$
$$H(C_3) = \psi_2 \otimes \Lambda^*([f\xi],[\eta]).$$

To compute C_1, define the complex

$$C_1' = \Lambda^*(s_1,s_2)\otimes \mathbb{C}[u_1,u_2]/(f^2,g^2) \quad \mathrm{d}s_j = u_j.$$

Then we have a short exact sequence of cochain complexes

$$0 \to \Lambda^*(s_1,s_2)\otimes(fg) \to C_1' \xrightarrow{\pi} C_1 \to 0$$

and as above

$$H(C_1') = \Lambda^*([f\xi],[g,\eta]).$$

Routine calculations with the long-exact sequence of the exact sequence above show that

$H(C_1)$ = generated by $\{[f\xi],[g\eta],[g\xi],[f\xi\wedge\eta],[g\xi\wedge\eta]\}$. The first two elements are in the image of π, and the last three come from the coboundary $H(C_1)\to H(\Lambda^*(s_1,s_2)\otimes(fg))$. Notice also that π has a 1-dimensional kernel generated by $[f\xi]\wedge[g\eta]$.

Hence

$$\sum_1^3 \dim H(C_j) = 13 \quad \text{and so} \quad \sum_{p,q} \dim E_2^{p,q} = 26. \qquad //$$

This computation gives the disproof of the folk-lore conjecture concerning the Hodge–de Rham spectral sequence, but just barely. The element $\theta \in E_1^{2,1} = E_2^{2,1}$ is the unique element of total degree 3 and hence a permanent cocycle. Considering the bidegrees of $H(C_j)$ one can work out easily enough that $d_2([f\xi\wedge\eta]) = \psi_1$, $d_2([g\xi\wedge\eta]) = \psi_2$ and that $d_r = 0$ for $r \geqslant 3$, because with the given formulae $\dim_{\mathbb{C}} E_3 = 16$.

We will now verify the conjecture 5.12 for the given structure on $SO(9)$. This will also complete our description of the ring structure of $H_{\bar\partial}(SO(9))$, the point being that with the bidegrees given $I_2 = I_\infty$, and hence by a dimension count that $II_2 = II_\infty$ (in the notation preceding 5.12). Thus the conjecture is true; and since the ring-structure is immediate in II_2, it follows that $f\psi_1 = 0$ and $\psi_1 \wedge \psi_2 = 0$. The reader who was uneasy about undertaking the proof of 6.5 without the full ring-structure of E_1 in hand, can do so after the discussion below, which is independent of 6.5.

So consider the complex $\Lambda^*(\varphi_1,\ldots,\varphi_4)\otimes\mathbb{C}[u_1,u_2]$ with $d\varphi_j = f_j|_{u_0}, du_j =$ Since $d\varphi_1 = 0$ the cohomology is the tensor product of $\Lambda^*(\varphi_1)$ and $H(\Lambda^*(\varphi_2,\varphi_3,\varphi_4)$ $\mathbb{C}[u_1,u_2]$). The formulae 6.1.1 show that modifying the φ_j by "unipotent" $\mathbb{C}[u_1,u_2]$ linear changes we can replace the exterior generators with $\zeta_j (2\leqslant j\leqslant 4)$ such that

$$d\zeta_2 = f^2,\quad d\zeta_3 = fg,\quad d\zeta_4 = g^2.$$

It follows that the cohomology of type (p,p) is precisely the ring $R^* = \mathbb{C}[u_1,u_2]$ (f^2, fg, g^2). Now define

$$\psi_1 = g\zeta_2 - f\zeta_3 \quad \psi_2 = f\zeta_4 - g\zeta_3.$$

Then $d(\zeta_2\wedge\zeta_3\wedge\zeta_4) = \pm\psi_1\wedge\psi_2$ and

$$d(\zeta_2\wedge\zeta_3) = -f\psi_1,\quad d(\zeta_2\wedge\zeta_4) = f\psi_2 - g\psi_1,\quad d(\zeta_3\wedge\zeta_4) = g\psi_2.$$

It follows that ψ_1,ψ_2 are closed and not exact, and

$$\psi_1\wedge\psi_2 = 0,\quad f\psi_1 = 0,\quad g\psi_2 = 0,\quad f\psi_2 = g\psi_1$$

in cohomology which easily yields the result of 6.5.2.

We end with a brief discussion of the kind of syzygies one can expect for a gene LICS on $SO(9)$. It is convenient to adopt a projective viewpoint here, and to repl two planes $\mathfrak{a}\subseteq\mathfrak{h}$ without real points with projective lines $\mathbb{P}(\mathfrak{a})\subseteq\mathbb{P}_3 = \mathbb{P}(\mathfrak{h})$ wh have no real points in the given homogeneous coordinates $[x_1,\ldots,x_4]$.

Consider the Fermat surfaces† in $\mathbb{P}_3$ defined as

$$S_m : \sum_1^4 x_j^{2m} = 0.$$

Any line which lies on these surfaces has no real points, and therefore corresp to an LICS on $SO(9)$. Our first result is that for $1\leqslant m\leqslant 4$ the only pair of surfa which share common lines are the quadric S_1 and the sextic S_3 and there are tw of these: indeed $S_1\cap S_3 =$ union of twelve lines which we will describe explicitly. now do some "high-school" geometry.

Let P_j be the co-ordinate plane in $\mathbb{P}_3$ defined by $x_j = 0$. A point (line) on P called *planar*. There are six biplanar lines $L_{ij} = P_i\cap P_j$ $(l<j)$ and points on th lines are called *biplanar*. Each L_{ij} intersects S_m in $2m$ distinct points correspond to the $2m$th roots of -1. Fix a biplanar point $\mathbf{a} = [i,\omega,0,0]$ on $S_m, \omega^{2m} = -1$. lines which lie on S_m and pass through $\mathbf{a}$ must also lie on the tangent plane to S_m $\mathbf{a}$; and hence the (parametric) coordinates of such a line must satisfy

$$x_1 + \omega^{2m-1}x_2 = 0.$$

It follows easily from this last fact that a line $L\subseteq S_m$ passing through $\mathbf{a}$ is uniqu determined by a "complementary" biplanar point on S_m of the form $[0,0,1,$

† Some of what we say below does not depend on the exponent being even, but the notation b conforms to our notation for $f_j\in\mathbb{C}[X_1,\ldots,X_4]$.

$(\omega')^{2m} = -1$; and any line determined by such a complementary pair of biplanar points on S_m lies entirely on S_m. There are $12m^2$ such lines on S_m, and we call them the *basic* lines on S_m. Of course, S_1 is doubly-ruled, but there are only 12 basic lines on it.

PROPOSITION 6.6

For $m \geqslant 2$ the only lines on S_m are the basic lines.

Proof. $P_j \cap S_m$ is a smooth, irreducible curve of degree $2m$, and hence a line $L \subseteq S_m$ cannot be planar. Thus $L \cap P_j$ is a single point, and there are precisely 2 distinct planar points on L, say $\mathbf{a}$ and $\mathbf{b}$, because $P_i \cap P_j \cap P_k$ cannot lie on any S_m. The parametric form of L is then $\lambda\mathbf{a} + \mu\mathbf{b}$ $(\lambda, \mu \in \mathbb{C})$ and since $L \subseteq S_m$,

$$\sum_{j=1}^{4} (\lambda a_j + \mu b_j)^{2m} = 0 \quad \text{for all } \lambda, \mu.$$

The vanishing of the coefficients of λ^{2m} (resp. μ^{2m}) says that $\mathbf{a} \in S_m$ (resp $\mathbf{b} \in S_m$). If $m \geqslant 2$ there are at least two other conditions to be satisfied

$$\sum_{1}^{4} a_j^{2m-1} b_j = 0 \quad \text{and} \quad \sum_{1}^{4} a_j^{2m-2} b_j^2 = 0.$$

Since $\mathbf{a}$ and $\mathbf{b}$ lie in distinct co-ordinate planes there are only two summands in each of the equations above and solving them we find that $\mathbf{a}, \mathbf{b}$ are biplanar points of S_m. But we have already determined all lines on S_m passing through biplanar points. //

It now follows from inspection that if $\{m_1, m_2\} \subseteq \{1, 2, 4\}$ or $\{m_1, m_2\} \subseteq \{2, 3, 4\}$ then there is no line on $S_{m_1} \cap S_{m_2}$. On the other hand since $\pm\sqrt{-1}$ is also a 6th root of -1 the twelve basic lines of S_1 lie on S_3: and since $S_1 \cap S_3$ is a connected curve of degree 12, $S_1 \cap S_3 =$ union of the 12 basic lines of S_1. (This can also be checked by elementary algebra.) Hence for our computation of $H^{*,*}(B)$ for $SO(9)$ we see that

(a) if $\mathfrak{a} \leftrightarrow$ one of the 12 basic lines, $f_1 = 0$ $f_3 = 0$
(b) if $\mathfrak{a}$ is any other line without real points, then at most *one* of the f_j can vanish.

In the first case, the twelve basic lines are permuted transitively amongst themselves under the (projective) linear action of the Weyl group W of $SO(9)$ on $\mathbb{P}_3 = \mathbb{P}(\mathfrak{h})$, and hence the complexes for these twelve cases are isomorphic. It follows easily that in these twelve cases

$$H_{\bar{\partial}}(SO(9)) \simeq \Lambda^*(\mathfrak{a}^*) \otimes \Lambda^*(\theta_1, \theta_2) \otimes \mathbb{C}[u, v]/u^4 + v^4, u^4 v^4),$$

where bideg $\theta_1 = (2, 1)$ and bideg $\theta_2 = (5, 6)$ and the Hodge de Rham sequence collapses at E_2.

In the second case, we have to understand the number of generators of the ideal $I_W(\mathfrak{a}) = (f_1, \ldots, f_4)$. For simplicity suppose the line is isotropic so that in any event $f_1 = 0$. If the number of generators of $(f_2, \ldots, f_4)$ is two-that is, if the points of intersection of $\mathbb{P}(\mathfrak{a})$ and S_2 or S_3 are among the intersection points of $\mathbb{P}(\mathfrak{a})$ with S_3 or S_4 then the resulting complex $B^{*,*}$ behaves like the first case (with a possible change in the bidegree of the second exterior generator) and again the Hodge de Rham sequence collapses at E_2. What makes our earlier example with $\mathfrak{a}_0$ work is

this. The line $\mathbb{P}(\mathfrak{a}_0)$ is isotropic (lies on S_1) and is bitangent to S_2 so that there are only 2 points of intersection in $\mathbb{P}(\mathfrak{a}_0)\cap S_2$. For any such $\mathbb{P}(\mathfrak{a})$ even if these two bitangent points are among the points of $\mathbb{P}(\mathfrak{a})\cap S_3$ they must occur with multiplicity one in this latter intersection: indeed, every line on S_1 (of type A or B) will intersect the sextic S_3 in six distinct points, corresponding to the six lines of type B or A on $S_1\cap S_3$. Hence f_3 can be zero divisor modulo f_2, but cannot contain it as a factor: $(f_2|_{\mathfrak{a}}, f_3|_{\mathfrak{a}}) \neq (f_2|_{\mathfrak{a}})$. If this is so, then the points of $\mathbb{P}(\mathfrak{a})\cap S_4$ must be disjoint from those of $\mathbb{P}(\mathfrak{a})\cap S_2$ or those of $\mathbb{P}(\mathfrak{a})\cap S_2$: if all these three sets had a common intersection, then the ring $R^* = S^*(\mathfrak{a}^*)/I_w(\mathfrak{a})$ would be infinite-dimensional, which is impossible, since $R^* \subseteq H_{\bar{\delta}}(SO(9))$. Thus the geometric crux of our example is that $\mathbb{P}(\mathfrak{a}_0)$ lies on S_1, is bitangent to S_2 and these two bitangent points occur among the six points of $\mathbb{P}(\mathfrak{a}_0)\cap S_3$. It is tempting to assert that the non-degeneracy of $H_{\bar{\delta}}(G)\Rightarrow H_{\mathrm{DR}}(G)$ is equivalent to the ideal $I_W(\mathfrak{a})$ having more than $(\dim \mathfrak{a})$ generators: and although I think it likely, I do not want to say it more forcefully.

7. The Atiyah–Chern classes of G

For every compact, complex manifold M and holomorphic vector bundle $V\to M$ Atiyah [1] has defined Chern classes in the "middle" Dolbeault ring $\oplus_p H^p(M,\Omega^p_M)$ which abut to the usual Chern classes under the Hodge–deRham spectral sequence. We shall calculate these classes for the holomorphic tangent bundle of G equipped with a fixed but arbitrary LICS: and we shall exhibit their dependence on moduli. Let us denote these refined Chern classes of $T'M$ by

$$ac_p(T'M) = ac_p(M)\in H^p(M,\Omega^p_M).$$

Recall from 2.7 that for any LICS on G there is a holomorphic principal-bundle

$$T\to G\xrightarrow{\pi} G/T \tag{7.1.1}$$

T being a maximal torus.

PROPOSITION 7.1

$ac_j(G) = \pi^* c_j(G/T)$

Proof. As (7.1.1) is a holomorphic principal bundle, the bundle of holomorphic tangent vectors along the fibres is trivial, and so we have a short-exact sequence of bundles over G

$$0\to\mathbb{C}^k\to T'G\to\pi^*T'(G/T)\to 0,$$

where $T'(M)$ = holomorphic tangent bundle of M. As Atiyah's classes satisfy the usual additivity condition, we have

$$ac_j(G) = \pi^*(ac_j(G/T))$$

and we only have to recall that G/T being Kahler the refined Chern classes coincide with the usual ones under the Dollbeault isomorphism. //

From the computations of § 5 one sees that $\pi^* H(G/T) \subseteq H_{\bar\partial}^{*,*}(G)$ is given by

$$\bigoplus_p H_0^p(\Lambda^p(m)) = S^*(\mathfrak{a}^*)/I_W(\mathfrak{a})$$

and we shall now say a few words about how the image $\pi^* c_j(G/T)$ can be calculated in this ring. The bundle $T'(G/T)$ over G/T (hence also over G) is the homogeneous vector bundle associated to the complex T-module $\mathfrak{g}^c/\mathfrak{g} \simeq \mathfrak{n}_-$; and hence $T'(G/T)$ is a successive extension of holomorphic line bundles corresponding to the negative roots. From the 'additivity' of Chern classes over extensions, one sees easily that

$$c_j(G/T) = j\text{th elementary symmetric function in } \{-\alpha, \alpha \in \Delta^+\}.$$

Now $H^*(G/T) = S^*(\mathfrak{h}^*)/I_W$, can be written out as

$$\mathbb{C}[\alpha_1, \ldots, \alpha_l]/I_W \quad \alpha_i \text{ the basic roots}$$

and the Chern classes can then be computed from the above data. To compute $\pi^* c_j(G/T)$ one has to write out co-ordinates for $\mathfrak{a}^*, \bar{\mathfrak{a}}^*$ in terms of $\alpha_1, \ldots, \alpha_l$: to rewrite $\pi^* c_j(G/T)$ on this basis, and then to drop the summands in the ideal $(\bar{\mathfrak{a}}^*)$.

PROPOSITION 7.2

$ac_1(G) \neq 0$.

Proof. $c_1(G/T) = \sum_\Delta \alpha$ and the projection of this element, which is real and non-zero in $\mathfrak{h}^*$ must be non-zero on $\mathfrak{a}^*$. //

In fact as we shall see below, the height of $ac_1(G)$ depends on moduli. Note that 7.2 shows again that $T'G \to G$ is holomorphically nontrivial.

Example 7.3. We will calculate $ac_j(SU(3))$ for any LICS on $SU(3)$. First of all, if α_1, α_2 are basic roots for $SU(3)$ then

$$H^*(SU(3))/T:\mathbb{C}) = \mathbb{C}[\alpha_1, \alpha_2]/(\alpha_1^2 + \alpha_2^2 + \alpha_1\alpha_2, \alpha_1^2(\alpha_1 + 2\alpha_2)).$$

Since the Chern classes of $SU(3)/T$ are the elementary symmetric functions in $\{-\alpha_1, -\alpha_2, -(\alpha_1 + \alpha_2)\}$ we have

$$c_1 = -2(\alpha_1 + \alpha_2), \quad c_2 = 2\alpha_1\alpha_2, \quad c_3 = -\alpha_1\alpha_2(\alpha_1 + \alpha_2),$$

where we have used the first element in I_W to rewrite c_2. Using the relations in I_W again we find that

$$c_1^2 = 2c_2, \quad c_1^3 = 4c_3.$$

Now we have already remarked that $\pi^* c_1 \in H^{1,1}(G)$ is not zero; and since this space is 1-dimensional, generated by $y_{1,1}$ (see 4.5).

$$\pi^* c_1 = \lambda y_{1,1} \quad \lambda \in \mathbb{C}^*.$$

Thus $\pi^* c_2, \pi^* c_3$ are proportional to $y_{1,1}^2$ and $y_{1,1}^3$: and again from 4.5.

$$\pi^* c_3 = 0$$

$\pi^* c_2 = 0$ in the anisotropic case, $\pi^* c_2 = \lambda^2 y_{1,1}^2$ in the isotropic case. //

Of course, the assertion $\pi^* c_3 = 0$ or more generally, $\pi^* c_{\text{top}} = 0$ is obvious from 5.8.

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eometry of the Mathieu groups and Golay codes

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Abstract. A brief review is given of the linear fractional subgroups of the Mathieu groups. The main part of the paper then deals with the projective interpretation of the Golay codes; these codes are shown to describe Coxeter's configuration in $PG(5,3)$ and Todd's configuration in $PG(11,2)$ when interpreted projectively. We obtain two twelve-dimensional representations of M_{24}. One is obtained as the collineation group that permutes the twelve special points in $PG(11,2)$; the other arises by interpreting geometrically the automorphism group of the binary Golay code. Both representations are reducible to eleven-dimensional representations of M_{24}.

Keywords. Geometry; Mathieu groups; Golay codes; Coxeter's configuration; hemi-icosahedron; octastigms; dodecastigms.

Introduction

shall first review some of the known properties of the Mathieu groups and their ear fractional subgroups. This is mainly a brief resumé of the first two of Conway's ree Lectures on Exceptional Groups' [3] and will serve to establish the notation and derlying concepts of the subsequent sections.

The six points of the projective line $PL(5)$ can be labelled by the marks of $GF(5)$ gether with a sixth symbol ∞ defined to be the inverse of 0. The symbol set is Ω $\{0\,1\,2\,3\,4\,\infty\}$. The group $L_2(5)$ of homographies on the line is generated by the dulo 5 transformations on Ω:

$$\alpha: i \to i+1, \quad \beta: i \to -i, \quad \gamma: i \to -1/i. \tag{1.1}$$

permutations,

$$\begin{aligned} \alpha &= (0\,1\,2\,3\,4) \\ \beta &= (14)(23) \\ \gamma &= (0\,\infty)(14). \end{aligned} \tag{1.2}$$

The group S_6 of all permutations of the six symbols is generated by α and τ, where

$$\tau = (0\,\infty). \tag{1.3}$$

a modulo 5 transformation, $\tau: i \to 1/i^3$). Alternatively, the S_6 is generated by α and π,

where

$$\pi = (0\,\infty)(14)(23) \tag{1.4}$$

($\pi: i \to -1/i^3$). The mapping θ on S_6 defined by

$$\alpha^\theta = \alpha \quad \tau^\theta = \pi \tag{1.5}$$

is an involutory outer automorphism of S_6. It is obviously not an inner automorphism because the fifteen transpositions correspond, under θ, to the fifteen regular involutions (products of three mutually commuting transpositions). These correspondences are the same as the correspondences between Sylvester's *duads* and *synthemes* of six symbols [1], [2], [6]. Any transposition appears in the structure of three of the regular involutions and each regular involution is comprised of three transpositions—a Cremona–Richmond 15_3. The self-dual property of the 15_3 corresponds to the involutory property of θ ($\theta^2 = 1$).

The S_6 can alternatively be generated by the five transpositions $(0\,\infty)$, $(1\,\infty)$ $(2\,\infty)$ $(3\,\infty)$, $(4\,\infty)$ or by the five regular involutions

$$\begin{aligned} \pi_0 &= (\infty\,0)(14)(23), \\ \pi_1 &= (\infty\,1)(20)(34), \\ \pi_2 &= (\infty\,2)(31)(40), \\ \pi_3 &= (\infty\,3)(42)(01), \\ \pi_4 &= (\infty\,4)(03)(12), \end{aligned} \tag{1.6}$$

(obtained by acting on π with the powers of α).

Under the action of θ, the generators α, β, γ of $L_2(5)$ are mapped to α, β, δ, where

$$\delta = (12)(34). \tag{1.7}$$

Whereas the original $L_2(5)$, generated by α, β, γ is transitive on the six symbols, the $L_2(5)$ generated by α, β, δ acts on only five of them. It is in fact the alternating group A_5 of even permutations of $\{0\,1\,2\,3\,4\}$. We have the isomorphism $L_2(5) \sim A_5$. The action of α, β, γ on the set Ω induces the action of α, β, δ on the set of five objects (1.6).

$$(\pi_i)^\alpha = \pi_{i\alpha}, \quad (\pi_i)^\beta = \pi_{i\beta}, \quad (\pi_i)^\gamma = \pi_{i\delta} \tag{1.8}$$

(the notation here is $\pi^\alpha = \alpha^{-1}\pi\alpha$, etc).

The four permutations α, β, γ and δ together generate the group A_6 of all even permutations of six symbols. Indeed, this A_6 is generated by α, β, and δ only, since $\beta = (\gamma\delta)^2$.

The S_6 together with its outer automorphism θ generates a group $S_6 2$ (a group with an invariant subgroup S_6 of index 2).

The eight points of the projective line $PL(7)$ can be labelled by the set $\Omega = \{0\,1\,2\,3\,4\,5\,6\,\infty\}$ consisting of the marks of $GF(7)$ and the extra symbol ∞. The group $L_2(7)$ of homographies on the line is generated by the modulo 7 transformations

$$\alpha: i \to i + 1, \quad \beta: i \to 2i, \quad \gamma: i \to -1/i. \tag{1.9}$$

As permutations,

$$\alpha = (0\,1\,2\,3\,4\,5\,6),$$
$$\beta = (124)(365),$$
$$\gamma = (0\,\infty)(16)(23)(45). \tag{1.10}$$

Analogously to the case of $PL(5)$, there exists another $L_2(7)$ that acts on only seven of the eight symbols; it is generated by α, β and

$$\delta = (12)(36). \tag{1.11}$$

The two groups $L_2(7)$ are mapped to each other by an involutory outer automorphism of the group F generated by α, β, γ and δ. In Figure 1, the vertices of Fano's configuration 7_3 have been labelled by the marks of $GF(7)$ (observe that α cyclically permutes the vertices of the 'self-inscribed and circumscribed heptagon' 0 1 2 3 4 5 6, and that β rotates the figure and δ reflects it). The projective plane $PG(2,2)$ is a Fano 7_3 and α, β and γ generate its group of collineations—the simple group $L_3(2)$ of order 168. We have the isomorphism $L_2(7) \sim L_3(2)$.

The quadratic residues of $GF(7)$ are the elements of the set $Q = \{0\,1\,2\,4\}$. The complementary set is $N = \{3\,5\,6\,\infty\}$. Consider the seven pairs of four symbols obtained by operating with the powers of α on Q, N:

$$\begin{aligned}
P_0 &= \{0\,1\,2\,4\}\{3\,5\,6\,\infty\},\\
P_1 &= \{1\,2\,3\,5\}\{4\,6\,0\,\infty\},\\
P_2 &= \{2\,3\,4\,6\}\{5\,0\,1\,\infty\},\\
P_3 &= \{3\,4\,5\,0\}\{6\,1\,2\,\infty\},\\
P_4 &= \{4\,5\,6\,1\}\{0\,2\,3\,\infty\},\\
P_5 &= \{5\,6\,0\,2\}\{1\,3\,4\,\infty\},\\
P_6 &= \{6\,0\,1\,3\}\{2\,4\,5\,\infty\}.
\end{aligned} \tag{1.12}$$

The action of the $L_2(7)$ generated by α, β, γ, on Ω, induces the action of the $L_2(7)$ generated α, β, δ on this set of seven objects;

$$\alpha: P_i \to P_{i\alpha}, \quad \beta: P_i \to P_{i\beta}, \quad \gamma: P_i \to P_{i\delta}. \tag{1.13}$$

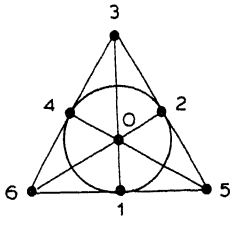


Figure 1.

The seven objects P_i are the seven pairs of complementary tetrads of a Steiner system $S(3,4,8)$. (Conway employs the involution $\pi = (\infty\,0)(13)(26)(45)$ instead of our object Q, N. The effect is the same. Our seven objects P_i are more suitable for our later purposes).

The twelve points of $PL(11)$ can be labelled by the symbol set $\Omega = \{0\,1\,2\,3\,4\,5\,6\,7\,8\,9\,X\,\infty\}$. The first eleven of these symbols are the marks of $GF(11)$. The homography group $L_2(11)$ is generated by

$$\alpha: i \to i+1, \quad \beta: i \to 3i, \quad \gamma: i \to -1/i \tag{1.14}$$

(modulo 11), i.e. by the permutations

$$\begin{aligned} \alpha &= (0\,1\,2\,3\,4\,5\,6\,7\,8\,9\,X), \\ \beta &= (1\,3\,9\,5\,4)(8\,2\,6\,7\,X), \\ \gamma &= (0\,\infty)(1X)(25)(37)(48)(69). \end{aligned} \tag{1.15}$$

The set $Q = \{0\,1\,3\,9\,5\,4\}$ is the set of quadratic residues. The non-zero quadratic residues are the powers of 3 modulo 11. The complementary set is $N = \{8\,2\,6\,7\,X\,\infty\}$. Observe that β leaves invariant these two hexads and γ interchanges them.

The Mathieu group M_{12} acting on the twelve symbols is generated by $\alpha, \beta, \gamma, \Delta$, where

$$\Delta = (2X)(34)(59)(67). \tag{1.16}$$

As is well known, M_{12} is the automorphism group of the Steiner system $S(5,6,12)$. The 66 complementary pairs of hexads are the images under the action of M_{12} of the generic hexad pair Q, N. Observe that $\beta = (\gamma\Delta)^6$ so in fact M_{12} is generated by α, γ, Δ. Our original subgroup $L_2(11)$ was generated by α, β, γ. Another $L_2(11)$ subgroup is generated by α, β, Δ. The original $L_2(11)$ is transitive on the 12 symbols while the other $L_2(11)$ acts on only 11 of them. This is analogous to the previous cases. However, the $L_2(11)$ generated by α, β, γ is maximal in M_{12}, while the $L_2(11)$ generated by α, β, Δ is a subgroup of M_{11}, so there can be no question of an automorphism of M_{12} mapping these two groups onto each another. On the other hand we have an automorphism θ defined by

$$\alpha^\theta = \alpha^{-1}, \quad \beta^\theta = \beta, \quad \gamma^\theta = \gamma, \quad \Delta^\theta = \Delta. \tag{1.17}$$

The 24 points of the projective line $PL(23)$ can be labelled by $\Omega = \{0\,1\,2\,3 \ldots 22\,\infty\}$ consisting of the marks of $GF(23)$ and the symbol ∞. The homography group $L_2(23)$ is generated by

$$\alpha: i \to i+1, \quad \beta: i \to 2i, \quad \gamma: i \to -1/i \tag{1.18}$$

(modulo 23). As permutations,

$$\begin{aligned} \alpha &= (0\,1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\,13\,14\,15\,16\,17\,18\,19\,20\,21\,22), \\ \beta &= (1\,2\,4\,8\,16\,9\,18\,13\,3\,6\,12)(5\,10\,20\,17\,11\,22\,21\,19\,15\,7\,14), \\ \gamma &= (0\,\infty)(1\,22)(2\,11)(4\,17)(8\,20)(16\,10)(9\,5)(18\,14)(13\,7)(3\,15)(6\,19)(12\,21) \end{aligned} \tag{1.19}$$

The group is extended to the Mathieu group M_{24} by the supplementary generator

$$\delta = (7\,5\,21\,20\,10)(22\,14\,17\,11\,19)(13\,9\,12\,8\,16)(1\,18\,4\,2\,6). \tag{1.20}$$

In fact M_{24} is generated by α, β, and $\gamma\delta$, since $(\gamma\delta)^5 = \gamma$ and $(\gamma\delta)^6 = \delta$.
We have

$$\gamma\delta = (\infty\,0)(3\,15)(1\,14\,4\,11\,6\,22\,18\,17\,2\,19)(5\,12\,20\,16\,7\,9\,21\,8\,10\,13). \tag{1.21}$$

The generators α, β, and δ generate M_{23} (the one-point stabilizer fixing the symbol ∞). M_{24} is the automorphism group of the Steiner system $S(5, 8, 24)$. The automorphism group is transitive on the 759 special octads of the Steiner system, and on the 2576 umbral dodecads (an umbral dodecad is constructed as the symmetric difference of two special octads with just two common symbols). Denoting the quadratic residue set by $Q = \{0\,1\,2\,4\,8\,16\,9\,18\,13\,3\,6\,12\}$ and the complementary set by N, then Q, N is a generic pair of complementary umbral dodecads of the $S(5, 8, 24)$. A complementary pair of umbral dodecads is called a 'duum'.

2. Correspondences

We have mentioned that S_6 is extended to a group $S_6 2$ by including the automorphism θ of S_6 as a supplementary generator. This group $S_6 2$ can be realized as a permutation group on twelve symbols. In fact, $S_6 2$ occurs in M_{12} as the stabilizer of a complementary hexad pair in $S(5, 6, 12)$. To see this, set up correspondences between $\Omega(5)$ and $Q(11)$, and between $\Omega(5)$ and $N(11)$, as follows

Ω:	0	1	2	3	4	∞
$Q(11)$:	1	3	9	5	4	0
$N(11)$:	8	2	6	7	X	∞

(2.1)

(i.e. $i \to 1/4^i$ and $i \to -3^{i+1}$ modulo 11). The permutations of S_6 on $\Omega(5)$ now induce permutations on $Q(11)$ and on $N(11)$. Consider the effect of acting on $Q(11)$ and $N(11)$ simultaneously with the two inequivalent representations of S_6. Thus the α of S_6 is realised on $\Omega(11)$ as the permutation

$$(1\,3\,9\,5\,4)(8\,2\,6\,7\,X), \tag{2.2}$$

which is just the β of M_{12}. The permutation τ of S_6 is realised on $\Omega(11)$ by the induced action of τ on $Q(11)$ and π on $N(11)$:

$$(8\,\infty)(10)(67)(2X). \tag{2.3}$$

This permutation also belongs to M_{12} (by a time-consuming process of trial and error and intuition, it was found to be Δ^η, where $\eta = \alpha^{-2}\gamma\alpha^3\beta^2$. There may be a simpler expression. A straightforward test for whether any particular permutation on $\Omega(11)$ belongs to M_{12} will be revealed in § 4). Finally, the automorphism θ of S_6 is realised on

$\Omega(11)$ by an interchange of $Q(11)$ and $N(11)$:

$$(18)(32)(96)(57)(4X)(0\,\infty). \tag{2.4}$$

This also belongs to M_{12}. As is easily verified, it is $(\gamma\Delta)^5$.

Incidentally: under the above correspondence, the β, γ and δ of S_6 induce, respectively, $\Delta, \Delta^{\alpha^{-3}}$ and $\Delta^{\alpha^{-3}(\alpha\Delta)^5}$ of M_{12}.

Analogously to the above, M_{12} can be extended to $M_{12}2$ by incorporating the automorphism θ of M_{12} as a supplementary generator, and this $M_{12}2$ occurs as a subgroup of M_{24} in the role of stabilizer of a duum of the Steiner system $S(5, 8, 24)$. To see how this works, set up the following correspondences $\Omega(11) \rightarrow Q(23)$ and $\Omega(11) \rightarrow N(23)$:

$\Omega(11)$:	0	1	2	3	4	5	6	7	8	9	X	∞
$Q(23)$:	3	13	18	9	16	8	4	2	1	12	6	0
$N(23)$:	15	7	14	5	10	20	17	11	22	21	19	∞

(2.5)

($i \rightarrow -2^{i+3}$ and $i \rightarrow 1/2^{i+3}$). Realize M_{12} as permutations on $\Omega(23)$ by inducing the actions of $\alpha^{-1}, \beta, \gamma, \Delta$ of M_{12} on $N(23)$, accompanied by the actions of $\alpha, \beta, \gamma, \Delta$ on $Q(23)$. Then the α and β of M_{12} induce, respectively, the permutations β and δ of M_{24}. The permutations induced by γ and Δ of M_{12} also belong to M_{24} (a simple test for this will be revealed later). The automorphism θ of M_{12} is realized as an interchange of $N(23)$ and $Q(23)$. It is in fact just the γ of M_{24}.

A set of three special octads of $S(5, 8, 24)$, with no symbol in common, is called a trio. $L_2(7)$ occurs as a subgroup of M_{12}, as a subgroup of the stabilizer of a trio. As a generic trio, we can take the three octads chosen by Curtis for the construction of the miracle octad generator (MOG) [4][5]. We set up correspondences between $\Omega(7)$ and each of these three octads, as follows:

$\Omega(7)$:	0	1	2	3	4	5	6	∞
	0	8	20	14	15	3	18	∞
	4	13	7	11	10	16	2	17
	9	1	12	22	6	5	21	19

(2.6)

Then the permutations α, β and γ of $L_2(7)$ induce permutations of $\Omega(23)$, which, as we shall show in § 6, belong to M_{24}. (Quite different $L_2(7)$ subgroups of M_{24} are the octern groups [3][5], which we shall not be concerned with in the present work).

3. The hemi-icosahedron and Coxeter's eleven-celled polytope

The group A_5 is the rotation group of an icosahedron. The representation of it as a transitive permutation group on *six* objects corresponds to the even permutations of the six diameters, brought about by the rotations. Its representation as the group of even permutations of five objects corresponds to the effect of the rotations on either a set of five inscribed cubes, or a set of five inscribed tetrahedra [8], or a set of five circumscribed octahedra [7], or a set of five trios of mutually perpendicular golden rectangles [7] (figure 3).

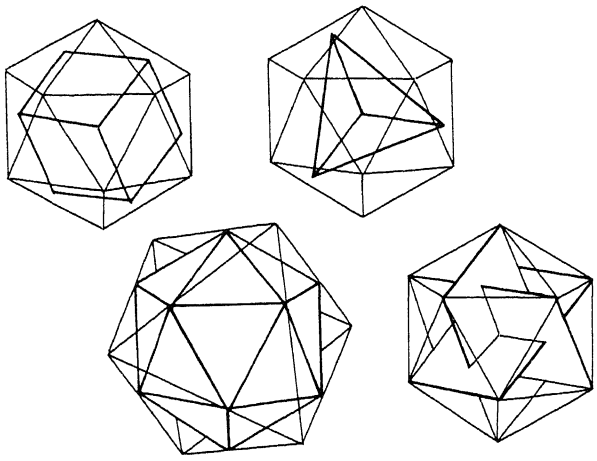


Figure 2.

The full symmetry group of the icosahedron, including reflections, can be generated by three reflections R_1, R_2, R_3 satisfying

$$R_1{}^2 = R_2{}^2 = R_3{}^3 = (R_2R_3)^2 = (R_1R_2)^3 = (R_1R_3)^5 = 1. \tag{3.1}$$

They can be chosen to be the reflections in the three diametral planes 36, 25 and 34 respectively (figure 3).

$$\begin{aligned} R_1 &= (15)(24)(ae)(bd) \\ R_2 &= (16)(34)(af)(ce), \\ R_3 &= (1f)(25)(a6)(be). \end{aligned} \tag{3.2}$$

The permutation $R_1R_2R_3 = (13245acbde)(6f)$ of order ten acts cyclically on the vertices of a Petrie polygon [8]. $(R_1R_2R_3)^5 = (1a)(2b)(3c)(4d)(5e)(6f)$ is the central inversion. The group can be restricted to the factor group A_5 by including $(R_1R_2R_3)^5 = 1$ as an extra generating relation. We then obtain generating relations for A_5 in the form

$$\begin{aligned} R_1{}^2 = R_2{}^2 = R_3{}^2 = (R_2R_3)^2 = (R_1R_2)^3 \\ = (R_1R_3)^5 = (R_1R_2R_3)^5 = 1. \end{aligned} \tag{3.3}$$

The generating relations for $L_2(5) \sim A_5$ given by Conway [3] are obtained immediately

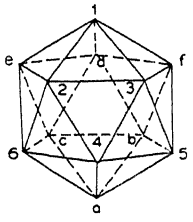


Figure 3.

by substituting

$$R_1 = \alpha\beta, \quad R_2 = \beta\gamma, \quad R_3 = \gamma. \tag{3.4}$$

That is, Conway's generators are given by

$$\alpha = R_1R_2R_3, \quad \beta = R_2R_3, \quad \gamma = R_3. \tag{3.5}$$

(The S, T and R of Todd [15] are respectively α, γ and $\beta\alpha$). The simplest set of generating relations for A_5 is that of the 'icosian calculus' of Hamilton [8], with only two generators ι and κ satisfying

$$\iota^2 = \kappa^3 = (\kappa\iota)^5 = 1. \tag{3.6}$$

They are obtained from (3.3) by substituting

$$\iota = R_2R_3, \quad \kappa = R_1R_2. \tag{3.7}$$

Employing (3.7), we find that $\kappa\iota\kappa^{-1}\iota = (R_1R_2R_3)^2$ and hence $(\kappa\iota\kappa^{-1}\iota)^3 = R_1R_2R_3$. We obtain then the inverse of the relations (3.7), in the form

$$\begin{aligned} R_1 &= (\kappa\iota\kappa^{-1}\iota)^3\iota \quad = (\kappa\iota\kappa^{-1}\iota)^2\kappa\iota\kappa^{-1}, \\ R_2 &= (\kappa\iota\kappa^{-1}\iota)^3\iota\kappa = (\kappa\iota\kappa^{-1}\iota)^2\kappa\iota, \\ R_3 &= (\kappa\iota\kappa^{-1}\iota)^3\iota\kappa\iota = (\kappa\iota\kappa^{-1}\iota)^2\kappa. \end{aligned} \tag{3.8}$$

It is time-consuming, but not difficult, to deduce all the relations (3.3) from (3.6) and (3.8).

The rotation group A_5 of an icosahedron is the symmetry group of a hemi-icosahedron (icosahedron with opposite points identified). The hemi-icosahedron has 6 vertices, 15 edges and 10 faces. Half-turns about mid-points of edges are not distinguishable from reflections, and central inversion is the identity. In figure 4, the six vertices have been labelled by the six symbols of $\Omega(5)$. The permutations

$$R_1 = (04)(13), \quad R_2 = (0\,\infty)(23), \quad R_3 = (0\,\infty)(14) \tag{3.9}$$

are reflections in the three diametral planes σ_1, σ_2, σ_3 through the edges 2∞, 14 and 23 respectively. Alternatively (and equivalently), they are half-turns about the mid-points of the edges 04, $0\,\infty$ and 14 respectively. It is easily verified that α, β and γ of $L_2(5) \sim A_5$ are obtained when (3.9) is substituted into (3.5). The five synthemes (1.6) correspond to the five sets of mutually perpendicular golden rectangles inscribed in an icosahedron.

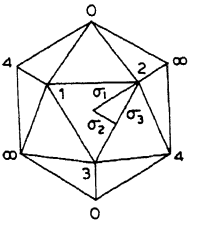


Figure 4.

n synthemes correspond to ten trios of golden rectangles like the one in ten trios are permuted by the rotations of the icosahedron.

ads of the Steiner system $S(5, 6, 12)$ on $\Omega(11)$ are obtained from the generic by applying the permutation α(1.15) repeatedly. These eleven hexads are long themselves by the $L_2(11)$ generated by α, β and Δ. The eleven hexads as sets of labels for the vertices of eleven hemi-icosahedra. In figure 6, the dron of figure 4 has been relabelled by the symbols of $Q(11)$, according to ndence $\Omega(5) \to Q(11)$ given in (2.1).

s hemi-icosahedron (0). Eleven hemi-icosahedra (i) $(i \in \Omega(11))$ are labelled belling of the vertices of $(i-1)$ is obtained from the labelling of the vertices ng one to each vertex label. We then find that, for any face of one of the dra, there is just one other hemi-icosahedron with a face labelled by the ymbols. By identifying vertices that carry the same symbol, the hemi- t together to form a polytope. This remarkable geometrical object was Grünbaum in 1976 [11] and independently by Coxeter in 1984 [9]. It has 5 edges, 55 faces and 11 cells. Its symmetry group is $L_2(11)$ and it is self-

oup A_5 that leaves the cell (0) invariant is generated by the three rmutations of $Q(11)$ induced by R_1, R_2 and R_3 under the correspondence

$_1 = (14)(35)(27)(8X)$,

$_2 = (10)(27)(59)(6X)$,

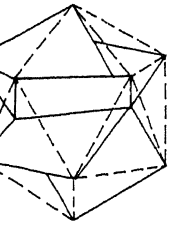
$_3 = (26)(7X)(10)(34)$.

permutations $\beta\Delta$, $\Delta^{\alpha\ 3}\Delta$ and $\Delta^{\alpha^{-3}}$. Observe that $S_1S_2S_3$ is β and S_2S_3 is Δ. $_4$ to be reflection in the 145 face. This interchanges the two cells (0) and

$_4 = (0X)(29)(36)(78)$.

erified, this is Δ^{α^4}. Finally, one can verify that $(S_1S_2S_3)^2\, S_1S_3S_4 = \alpha$. Thus d S_4 generate $L_2(11)$.

ality of the polytope corresponds to the automorphism θ of $L_2(11)$ (1.17). tions α, β, Δ of the eleven vertices correspond, respectively, to the α^{-1}, β and Δ of the eleven cells.



gure 5.

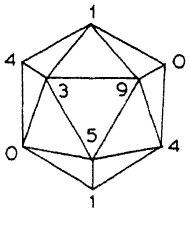


Figure 6.

4. The ternary Golay code

Let $\mathscr{V}$ be the vector space of 12-tuples over $GF(3)$, with a length measure imposed; the 12-tuples will be called *words* and the *weight* of a word is the number of non-zero components. The set $\Omega(11)$ can be employed as an index set for labelling the canonical base vectors (words of weight 1) $v_i\,(i\in\Omega)$. We find it convenient to let the index $i\in\Omega$ run through the values in the order

$$8\,2\,6\,7\,X\,0\,1\,3\,9\,5\,4\,\infty. \tag{4.1}$$

The *extended ternary Golay code* is the subspace $\mathscr{W}$ of $\mathscr{V}$ spanned by the rows $w_i\,(i\in\Omega)$ of the 12×12 matrix W defined by

$$W_{ij}=\begin{cases} 1, & i+j\in N \\ -1, & i+j\in Q \end{cases}. \tag{4.2}$$

With the canonical ordering (4.1) for the row and column labels,

$$W=\begin{bmatrix} X & j & Y & j \\ j^T & -1 & -j^T & 1 \\ Y & -j & -X & j \\ j^T & 1 & j^T & 1 \end{bmatrix}, \tag{4.3}$$

where j is a column of five 1s and X and Y are the 5×5 matrices $X=\mathrm{circ}(-1\,1-1-1\,1)$, $Y=\mathrm{circ}(-1-1\,1\,1-1)$ (i.e. the first rows of the matrices are as indicated and each row is obtained from the row above by cyclically shifting one place *to the right*). The elements of $\mathscr{W}$ (12-tuples consisting of linear combinations modulo 3 of the rows of W) are called *codewords*.

The code contains 3^6 words; the word of zero weight, 264 of weight 6, 440 of weight 9 and 24 of weight 12 ($1+264+440+24=729=3^6$). They can be characterized as the following particular linear combinations:

$$\begin{aligned} &\pm w_i \text{ are the 24 codewords of weight 12} \\ &\pm(w_i+w_j\pm w_k) \text{ are the 440 codewords of weight 9} \\ &\pm(w_i\pm w_j) \text{ are the 264 codewords of weight 6.} \end{aligned} \tag{4.4}$$

The effect on the w_i of a change of basis in $\mathscr{V}$ is obtained by multiplication of W *on the right* by a nonsingular matrix. Multiplying by

$$S=\begin{bmatrix} -Y & j & & \\ j^T & 1 & & \\ & & Y & j \\ & & j^T & -1 \end{bmatrix} \tag{4.5}$$

gives

$$W'=\begin{bmatrix} & & -K & j \\ & I_6 & -j^T & 0 \\ K & j & I_6 & \\ -j^T & 0 & & \end{bmatrix} \tag{4.6}$$

where I_6 is the unit 6×6 matrix and $K = \text{circ}(0\,1\,-1\,-1\,1\,1)$. We see immediately from the form of W' that $\mathscr{W}$ is 6-dimensional; denoting the ordered index sets 8267X0 and 13954∞ by $\bar{N}$ and $\bar{Q}$ respectively, we see that either $w_i(i\in\bar{N})$ or $w_i(i\in\bar{Q})$ provide a basis for $\mathscr{W}$. The components of the second basis with respect to the first are given by the rows of $\begin{pmatrix} K & j \\ -j & 0 \end{pmatrix}$, i.e.:

$$\begin{array}{c} 1 \\ 3 \\ 9 \\ 5 \\ 4 \\ \infty \end{array} \begin{bmatrix} 0 & 1 & -1 & -1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ -1 & 1 & 0 & 1 & -1 & 1 \\ -1 & -1 & 1 & 0 & 1 & 1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & 0 \end{bmatrix} \tag{4.7}$$

Of course, we have not preserved weights in going from W to W', so we cannot infer that the rows of W' span $\mathscr{W}$. On the other hand, by multiplication *on the left* by S, we get

$$U = \begin{bmatrix} & & K & -j \\ & I_6 & j^T & 0 \\ -K & -j & & I_6 \\ j^T & 0 & & \end{bmatrix}. \tag{4.8}$$

Either the first six rows of U, $u_i(i\in\bar{N})$, or the last six rows, $u_i(i\in\bar{Q})$, provide an alternative basis for the code.

The first of these is essentially the basis given originally by Golay [10].

The automorphism group of the Golay code consists of the linear transformations on $\mathscr{V}$ that preserve $\mathscr{W}$ and the weights of codewords. It is a binary Mathieu group $2M_{12}$ generated by the permutations

$$\begin{aligned} &A: v_i \to v_{i\alpha}, \quad B: v_i \to v_{i\beta}, \\ &C: v_i \to \pm v_{i\gamma}, \quad D: v_i \to v_{i\Delta}, \end{aligned} \tag{4.9}$$

of the 24 words of length 1, $\pm v_i$. The sign in C is $+$ for $i\in Q$ and $-$ for $i\in N$. The effect on the $w_i(i\in\Omega)$ is

$$\begin{aligned} &A: w_i \to w_{i\alpha}, \quad B: w_i \to w_{i\beta}, \\ &C: w_i \to \mp w_{i\gamma}, \quad D: w_i \to w_{i\Delta}, \end{aligned} \tag{4.10}$$

(where the sign in C is $-$ for $i\in Q$ and $+$ for $i\in N$). This remarkable result is due to Conway [3].

We now interpret the words projectively, as sets of homogeneous coordinates of points in a $PG(11, 3)$, which we call $\bar{\mathscr{V}}$. The code words are the sets of homogeneous coordinates of a subspace $PG(5, 3)$ which we call $\bar{\mathscr{W}}$. The twelve points $w_i(i\in\Omega)$ are thus twelve points in $PG(5, 3)$ whose coordinates are given by the rows of I_6 and the rows of the matrix (4.7). They are thus just the 12 points of Coxeter's configuration, which was shown by Coxeter[6] to have an automorphism group M_{12} consisting of collineations.

The correspondence between our notation and Coxeter's is:

$$\begin{array}{cccccccccccc} w_8 & w_2 & w_6 & w_7 & w_X & w_0 & w_1 & w_3 & w_9 & w_3 & w_4 & w_\infty \\ 1 & 2 & 3 & 4 & 5 & f & a & b & c & d & e & 6 \end{array}. \tag{4.11}$$

Interpreted as collineations in $PG(5,3)$, the transformations (4.10) are the collineations given by the matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.12}$$

(Check: multiply I_6 and the matrix (4.7) on the right!). They generate the group M_{12} of collineations that permute the 12 points.

The matrix W is symmetric and satisfies $W^2 = 0$. It follows immediately that $\bar{\mathscr{W}}$ lies entirely in the quadric $x_i x_i = 0$ $(i \in \Omega)$ in $\bar{\mathscr{V}}$ and that $\bar{\mathscr{W}}$ coincides with its polar 5-space. Thus, a point of $\bar{\mathscr{V}}$ is in $\bar{\mathscr{W}}$ if and only if it lies in the polar primes of all the twelve points $w_i (i \in \Omega)$. Therefore, a word with components $c_i (i \in \Omega)$ is a codeword if and only if $c_i w_i = 0$. This establishes a one-to-one correspondence between points in $\bar{\mathscr{W}}$ and syzygies satisfied by the twelve points (a syzygy is an equation expressing the vanishing of a linear combination of the w_i). Since there are 264 codewords of weight 6, there will be $264/2 = 132$ syzygies of length 6 satisfied by the coordinate of the twelve points of w_i. Coxeter lists them all explicitly. Each syzygy of length 6 expresses the condition that 6 points lie in a 4-space. Thus, the 12 points lie in sixes on 132 primes (hyperplanes) of the $PG(5,3)$. Each hexad of points is one of the hexads of the Steiner system $S(5,6,12)$.

It is useful at this stage to give a brief outline of the results of Coxeter's investigation of the configuration of 12 points in $PG(5,3)$.

A *hexastigm* is a set of six general points in a projective 4-space. A pair of *Möbius simplexes* in a projective space is a pair with the property that each vertex of one lies in a prime face of the other. The twelve points in $PG(5,3)$ can be regarded as a pair of complementary hexastigms in 66 ways, or as a pair of Möbius simplexes in 395 ways. The automorphism group M_{12} can be generated by 395 involutory collineations, each of which interchanges the simplexes of a Möbius pair. The coordinates of the points *ABCDEF* of a hexastigm can be chosen (by adjusting irrelevant factors) to satisfy a

syzygy

$$A+B+C+D+E+F=0. \tag{4.13}$$

One can then define 40 associated points as follows:
15 'harmonic' or negative points such as $A-B$ etc. 15 'positive' points such as $A+B-C-D\,(=C+D-E-F=E+F-A-B)$ etc. 10 'minor' points such as $A+B+C\,(=-D-E-F)$ etc. If the underlying field of the projective space in which the hexastigm lies is $GF(3)$, the 40 associated points all lie in a 3-space (since they are all obtainable as linear combinations of, for example, $A-F$, $B-F$, $C-F$ and $D-F$). Moreover, they are *all* of the points of the 3-space in which they lie ($40=(3^4-1)/2$). The negative points can be denoted by *duads* such as (AB) and the positive points by synthemes such as $(AB.\,CD.\,EF)$. The minor points may be denoted by $P_{ABC}\,(=P_{DEF})$ etc. The associated 3-spaces of a pair of complementary hexastigms coincide. More precisely, the negative points of each coincide with the positive points of the other, and the minor points of one coincide with minor points of the other. Each of these coincidences is indicated by several of the 132 syzygies. For example, consider Coxeter's two generic hexastigms 123456 and *abcdef* (in our notation, $w_i(i\in N)$ and $w_i(i\in Q)$). The coincidence $(16)=(af.\,cd.\,eb)$ is seen in the three syzygies

$6+c+d=1+e+b, 6+e+b=1+a+f, 6+a+f=1+c+d$ of Coxeter's list, and $P_{123}=P_{aef}$ is seen in the four syzygies $1+2+3+a+e+f=0, 4+5+6=a+e+f$, $1+2+3=b+c+d, 4+5+6+b+c+d=0$. The two generic syzygies $1+2+3+4+5+6=0, a+b+c+d+e+f=0$ together with all the syzygies associated with the coincidences of associated points, account for the whole list; $2+4\cdot10+3\cdot15+3\cdot15=132$.

The coincidences between negative points of 123456 and positive points of *abcdef* establish a one-to-one correspondence between number duads and letter synthemes. The table of duads and synthemes obtained from this one by the permutation $(1a)\,(2b)\,(3c)\,(4d)\,(5e)\,(6f)$ lists the coincidences between negative points of 123456 and positive points of *abcdef*. The permutation interchanges these two hexastigms and also the two simplexes 12345*f*, *abcde*6 of a Möbius pair. It corresponds to an involutary collineation in M_{12}. In our notation, it is the collineation with matrix $(CD)^5$ (see (2.4) and (4.11)). This matrix is (4.7) with the sign of the final row changed.

Of the $(3^6-1)/2=364$ points of $PG(5,3)$, twelve are the points w_i, 132 occur as positive or negative points and 220 occur as minor points ($12+132+220=364$). The positive or negative points are the $w_i\pm w_j$ which arise from the codewords of weight 6 when the Golay code is interpreted projectively, and the minor points are the $w_i+w_j\pm w_k$ which arise from the codewords of weight 9. The twelve points w_i of course arise from the words of weight 12.

The words of weight 1 can be interpreted projectively as the coordinates of twelve primes in $\bar{\mathscr{V}}$ (prime faces of the reference simplex). We call these twelve primes $\pi_i(i\in\Omega)$. Consider the intersection of the primes π_i by the 5-space $\bar{\mathscr{W}}$. We get twelve primes in $\bar{\mathscr{W}}$, which will also be called π_i. Referred to the coordinate system that we have established in $\bar{\mathscr{W}}$ (with $w_i(i\in\bar{N})$ as reference simplex), the coordinates of the twelve primes $\pi_i(i\in\Omega)$ are given (in the order (4.1)) by the *columns* of the matrices

$$\begin{bmatrix} X & j \\ j^T & -1 \end{bmatrix}, \quad \begin{bmatrix} Y & j \\ -j^T & 1 \end{bmatrix} \tag{4.14}$$

(Proof: Let c be a sextuple of homogeneous coordinates of a point in $\bar{\mathscr{W}}$. The components of this point, referred to the coordinate system of $\bar{\mathscr{V}}$, are given by the 12-tuple obtained by multiplying the row c by

$$\begin{pmatrix} X & j & Y & j \\ j^T & -1 & -j^T & 1 \end{pmatrix},$$

since this array is just the upper half of W and gives the coordinates of the base points $w_i (i \in \bar{N})$, of $\bar{\mathscr{W}}$. The condition for a point to lie on π_i is that the ith component of its 12-tuple shall vanish. Therefore, the prime π_i of $\bar{\mathscr{W}}$ is given by the ith column of the above array).

Clearly, the 12 primes π_i of $\bar{\mathscr{W}}$ are the 12 primes introduced into Coxeter's configuration by Todd [16]. The correspondence between our notation and Todd's is the following:

$$\begin{array}{cccccccccccc} \pi_8 & \pi_2 & \pi_6 & \pi_7 & \pi_X & \pi_0 & \pi_1 & \pi_3 & \pi_9 & \pi_5 & \pi_4 & \pi_\infty \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{e} & \mathbf{-6} & \mathbf{-1} & \mathbf{-2} & \mathbf{-3} & \mathbf{-4} & \mathbf{-5} & \boldsymbol{f} \end{array}. \tag{4.15}$$

Multiplying the two matrices (4.13) on the left by $M^{-1} = \begin{pmatrix} X & j \\ j^T & -1 \end{pmatrix}$ transforms them to $\begin{pmatrix} K & j \\ j^T & 0 \end{pmatrix}$ and $\begin{pmatrix} -\mathrm{I}_5 & 0 \\ 0 & 1 \end{pmatrix}$ respectively. The columns of these matrices, taken in order, are Coxeter's coordinate sextets $a, b, c, d, e, -6, -1, -2, -3, -4, -5, f$. Thus, the twelve primes are the polars of the twelve points, with respect to the quadric whose matrix is $M = \begin{pmatrix} -Y & j \\ j^T & 1 \end{pmatrix}$. The polars of Coxeter's 123456*abcdef* are, respectively, Todd's **123456abcdef.** There are 95040 quadrics with the property that the twelve primes are the polars of the twelve points (obtained from M by multiplying on the right by the transpose of any matrix of the collineation group M_{12}). A different quadric, more suitable to our present notation, is the quadric with matrix $Q = \begin{pmatrix} Y & j \\ -j^T & 1 \end{pmatrix}$ Multiplying the matrices (4.7) on the left by $Q^{-1} = \begin{pmatrix} -X & -j \\ -j^T & -1 \end{pmatrix}$ gives, respectively, I_6 and the transpose of the matrix (4.7). Thus, with respect to this quadric, the polarity is $\pi_i \leftrightarrow w_i (i \in \Omega)$.

The polarities show that the configuration of 12 points and 12 primes in $PG(5, 3)$ is self-dual. One can immediately infer the dual of all the properties of Coxeter's configuration: the 12 primes pass in sixes through 132 points and can be regarded in 395 ways as a pair of Möbius simplexes. Given any 'dual hexastigm' (six primes through a point), one can construct 40 associated primes: 15 'negative' primes, 15 'positive' primes and 10 'minor' primes. The associated primes of two complementary dual hexastigms coincide.

The 364 primes of $PG(5, 3)$ can be classified as follows. Twelve of them are the π_i, arising from the words of weight 1 of $\mathscr{V}$. 132 of them occur as 'positive or negative' primes; these arise from the words of weight 2, $\pi_i \pm \pi_j$. And 220 of them occur as 'minor' primes; these arise from the words of weight 3, $\pi_i + \pi_j \pm \pi_k$.

In $\mathscr{V}$, the words of weight m have $12 - m$ zero components. Hence, in $\bar{\mathscr{V}}$, a point whose coordinate 12-tuple is a word of weight m lies on just $12 - m$ of the primes π_i. In particular,

in $\mathscr{W}$ the twelve points w_i lie on none of the twelve primes π_i, the 220 'minor' points each lie on just 3 of the primes π_i, and the 132 'positive and negative' points each lie on just 6 of the primes π_i. The dual of this latter statement is: the 132 'positive and negative' primes each contain six of the twelve points w_i. Thus *the primes in which Coxeter's* 132 *hexastigms lie are the 'positive and negative' primes* $\pi_i \pm \pi_j$. In fact, the two primes $\pi_i \pm \pi_j$ (for fixed i and j) are the primes of a complementary pair of hexads.

Given any hexastigm of the points w_i, one can associate with it, according to the above, two of the primes α_i. We get 132 *octads* consisting of six of the points and two of the primes. Dually we get 132 octads consisting of two of the points and six of the primes. We can also form 495 octads consisting of four points and four primes, as follows. There are $\binom{12}{4} = 495$ ways of choosing four of the twelve primes. Going over to the $PG(11, 3)$ in which these four primes are specified by words of weight one, we can see by studying the rows of W that there are just four 'positive and negative' points that lie on all four of the primes. For example, π_8, π_2, π_6 and π_7 all contain $w_2 + w_6, w_x + w_4, w_0 - w_\infty, w_1 + w_5$. Just four of the symbols w_i have not appeared, namely w_8, w_7, w_3 and w_4. So we can specify an octad $\pi_8\pi_2\pi_6\pi_7 w_8 w_7 w_3 w_4$. The total number of octads is now $132 + 132 + 495 = 759$. Todd [16] shows that these octads are the octads of a Steiner system $S(5, 8, 24)$ on the 24 objects π_i, w_i.

5. The binary Golay code

Let $\mathscr{V}$ now denote a vector space of 24-tuples over $GF(2)$ with a length measure imposed on it; the *weight* of a 24-tuple is the number of non-zero elements. The 24-tuples will be called *words*. The set $\Omega(23)$ can be used as an index set labelling the base vectors v_i ($i \in \Omega$) and we let the index i run through Ω in the order

$$\begin{array}{cccccccccccc} 5 & 10 & 20 & 17 & 11 & 22 & 21 & 19 & 15 & 7 & 14 & 0 \\ 1 & 2 & 4 & 8 & 16 & 9 & 18 & 13 & 3 & 6 & 12 & \infty \end{array} \tag{5.1}$$

(5 times the powers of 2 modulo 23, in order, followed by zero, followed by the powers of 2, and finally the symbol ∞). We shall call the ordered set consisting of the first twelve of these symbols $\bar{N}$, and the ordered set consisting of the last twelve will be called $\bar{Q}$. The *extended binary Golay code* is the subspace $\mathscr{G}$ spanned by the rows N_i ($i \in \Omega$) of the matrix N defined by

$$N_{ij} = \begin{cases} 1, & i + j \in N(23) \\ 0, & i + j \in Q(23). \end{cases} \tag{5.2}$$

With the canonical ordering of the row and column labels,

$$N = \begin{bmatrix} X & j & Y & j \\ j^T & 0 & 0 & 1 \\ Y^T & 0 & Z & j \\ j^T & 1 & j^T & 1 \end{bmatrix} \tag{5.3}$$

where j is a column of 1s, 0 is the zero element or a row or column of eleven 0s,

depending on the context, and X, Y and Z are the 11×11 matrices

$$X = \text{circ } (11010000101),$$
$$Y = \text{circ } (01001100011),$$
$$Z = \text{circ } (00101111010). \tag{5.4}$$

The 12-tuples of $\mathscr{C}$ are called *codewords*. There are 2^{12} codewords; one of weight zero, 759 of length 8, 2576 of weight 12, 759 of weight 16 and 1 of weight 24 ($1 + 759 + 2576 + 759 + 1 = 4096 = 2^{12}$). Observe that the rows of the matrix N are 23 words of weight 12 and one of weight 24 (N_∞). Thus there is a fundamental asymmetry that was not present in the case of the ternary code—the rows of W were all of the same weight. This distinction turns out to be crucially important.

Changing the basis in $\mathscr{V}$ multiplies N *on the right* by a non-singular matrix. Multiplying by

$$S = \begin{bmatrix} P & j & & \\ j^T & 1 & & \\ & & Q & j \\ & & j^T & 0 \end{bmatrix}, \tag{5.5}$$

where $P = \text{circ } (01100000011)$ and $Q = \text{circ } (10011111100)$, gives

$$N' = \begin{bmatrix} & I_{12} & L & j \\ & & j^T & 0 \\ L^T & j & I_{12} & \\ j^T & 0 & & \end{bmatrix} \tag{5.6}$$

where

$$L = \text{circ } (11101110010). \tag{5.7}$$

We see from the form of N' that $\mathscr{C}$ is 12-dimensional, and that either the first twelve rows of $N(N_i, i \in \bar{N})$ or the last twelve rows $(N_i, i \in \bar{Q})$ provide a basis for $\mathscr{C}$.

An alternative pair of bases for $\mathscr{C}$ is obtained by multiplying N *on the left* by S. We get the same matrix N'(5.5), but the interpretation is different. We establish in this way that either the first twelve rows of N' or the last twelve rows of N' provide a basis for $\mathscr{C}$ (this was *not* established by the previous derivation of N'!). We shall call the rows of N', $u_i(i \in \Omega)$. The basis $u_i(i \in \Omega)$, consisting of the rows of the array

$$\begin{pmatrix} I_{12} & L & j \\ & j^T & 0 \end{pmatrix} \tag{5.8}$$

is the usual basis for the code. This elegant characterization of the code was discovered by Karlin [12]. Golay's original characterization [10] had a different matrix in place of L, related to L by permutations of the rows and columns. It is remarkable that the circulant L in Karlin's construction of the code is related to the quadratic residues modulo 11:The first row of L, (11011100010), is obtained from the set $\Omega'(11) = 0123456789X$ by replacing each symbol by 0 if it is in $N(11)$ and by 1 if it in $Q(11)$. The array (5.7) is essentially the array employed by Leech [13] in the construction of his celebrated lattice [18].

Omitting the null word, $\mathscr{C}$ can be interpreted projectively as a subspace $PG(11,2)$ (which we call $\bar{\mathscr{C}}$) of a $PG(23,2)$ (which we call $\bar{\mathscr{V}}$). Choosing the points $N_i(i\in\bar{Q})$ as the vertices of the reference simplex in $\bar{\mathscr{C}}$, the homogeneous coordinates of the other twelve points $N_i(i\in\bar{N})$ are given by the rows of $\begin{pmatrix} L & j \\ j^T & 0 \end{pmatrix}$, i.e.

$$\begin{array}{c|cccccccccccc}
5 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
10 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
20 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
17 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
11 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
22 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
21 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
19 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
15 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
7 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
14 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{array} \tag{5.9}$$

The 24 points $N_i(i\in\Omega)$ are the 24 points in $PG(11,2)$ studied by Todd [16][17], which are permuted by a group M_{24} of collineations.

The automorphism group of the binary Golay code is an M_{24} of permutations of the words of length 1. It is generated by the three permutations

$$A: v_i \to v_{i\alpha}, \quad B: v_i \to v_{i\beta}, \quad H: v_i \to v_{i\gamma\delta}, \tag{5.10}$$

where α, β and $\gamma\delta$ are the permutations of $\Omega(23)$ given in (1.19) and (1.21). The effect of A and B on the $N_i(i\in\Omega)$ is found to be given by the permutations

$$A: N_i \to N_{i\alpha^{-1}}, \quad B: N_i \to N_{i\beta} \tag{5.11}$$

but H does not permute them. The word of length 24, N_∞, remains fixed, but the other 23 are mapped to 23 different words. The images of the N_i are, however, still in the subspace $\bar{\mathscr{C}}$.

When this M_{24} is interpreted as a group of collineations in $\bar{\mathscr{C}}$, we find that A, B and H are the collineations with matrices given by

$$A = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},$$

$$B = \begin{bmatrix}
0 & 1 & & & & & & & & & & \\
0 & 0 & 1 & & & & & & & & & \\
0 & 0 & 0 & 1 & & & & & & & & \\
0 & 0 & 0 & 0 & 1 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 1 & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},$$

$$H = \begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}. \qquad (5.12)$$

That the collineations A and B have the correct action (5.11) on the 24 points $N_i (i \in \Omega)$ is easily checked: the coordinates of these 24 points of $\bar{\mathscr{C}}$ are the rows of (5.9) and the unit matrix, so their images under the action of A and B are given by the rows of the matrices given by multiplying (5.9) and I_{12} on the right of the matrices A and B.

The matrix H was obtained as follows: from the matrix N (5.9) we see that $N_1 = v\{10\,20\,21\,19\,14\,4\,16\,9\,18\,13\,6\,\infty\}$ (the notation vS means $\sum v_i (i \in S)$, where S is any subset of Ω). Acting on this with the transformation H given by (5.10) gives the image of N_1, which is $v\{13\,16\,18\,14\,11\,7\,21\,17\,5\,22\,0\}$. In terms of the basis $u_i (i \in \bar{Q})$ given by the last six rows of N' (5.6), this must be $u\{13\,16\,8\,1\,4\}$ (which can be verified by actually adding these six u_i, thus establishing that the image of N_1 really is in $\bar{\mathscr{C}}$). Finally we go over to the basis $N_i (i \in \bar{Q})$ which is related to the basis $u_i (i \in \bar{Q})$ by the matrix $\begin{pmatrix} Q & j \\ j^T & 0 \end{pmatrix}$ appearing in (5.5). We find $u\{13\,16\,8\,1\,4\} = N\{8\,16\,18\,13\,6\,\infty\}$. This establishes the first row of the matrix H. (The process sounds complicated when described but in practice is quite rapid).

Observe the curious structure of the matrix H. The 11×11 submatrix obtained by omitting the final row and final column has the property that each row is obtained from the previous row by cyclically shifting three places to the left. The explanation for this property is as follows. The α and β of M_{12} satisfy the generating relation $\alpha^\beta = \alpha^3$ [3].

Recall that they correspond to the β and δ of M_{24} under the embedding of M_{12} in M_{24} discussed in § 2. Therefore, the β and δ of M_{24} satisfy $\beta^{\delta} = \beta^3$. Together with the M_{24} generating relation $(\beta\gamma)^2 = 1$, this gives $\beta(\gamma\delta) = (\gamma\delta)\beta^{-3}$. Hence $BH = HB^{-3}$, which is the structural property of H that we noticed.

Since the point N_∞ remains invariant under the group M_{24} of collineations, the 12-dimensional representation of M_{24} generated by A, B and H is *reducible*. An *eleven-dimensional representation* of M_{24} is obtained by removing the final rows and columns from the matrices A, B and H. (In the projective geometry of $\bar{\mathscr{C}}$ this corresponds to projection from the point N_∞ (00...01) to the prime [00...01].

Presumably, the 11-dimensional representation of M_{23} discovered by Paige [14] is equivalent to the one generated by the reduced matrices A and B together with a matrix D obtained by omitting the last row and column from H^6. These three matrices represent the generators α, β, δ of the M_{23} that fixes v_∞. We find

$$D = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{5.14}$$

A different 12-dimensional representation of M_{24} is obtained by considering the collineations on $\bar{\mathscr{C}}$ which permute the 24 points $N_i (i \in \Omega)$: This is generated by the A and B of (5.12) together with

$$F: N_i \to N_{i\gamma\delta}. \tag{5.15}$$

The matrix of the collineation F on $\bar{\mathscr{C}}$ is

$$F = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \tag{5.16}$$

as is easily verified by multiplying I_{12} and the matrix (5.9) on the right.

6. Octastigms and dodecastigms

An *octastigm* may be defined as a set of eight general points in a projective 6-space. The coordinate sets of the eight points of an octastigm $ABCDEFGH$ can be chosen to satisfy a syzygy

$$A+B+C+D+E+F+G+H=0. \tag{6.1}$$

Associated with an octastigm are $\binom{8}{2}=28$ *duadic* points $A+B$ etc., and $\frac{1}{2}\binom{8}{4}=35$ *tetradic* points $A+B+C+D$ etc. In a projective space over $GF(2)$, these points associated with an octastigm all lie in a $PG(5,2)$ (their coordinate sets can all be obtained as linear combinations of those of six of the duadic points such as $A+H$, $B+H, C+H, D+H, E+H, F+H$); indeed they are all the points of the $PG(5,2)$ in which they lie ($28+35=63=2^6-1$).

The tetradic points associated with an octastigm in a projective space over $GF(2)$ lie in sevens on thirty planes. For example, the seven points

$$\begin{aligned}
&A+D+G+H=B+C+E+F,\\
&B+E+G+H=C+A+F+D,\\
&C+F+G+H=A+B+D+E,\\
&D+B+C+H=A+E+F+G,\\
&E+C+A+H=B+F+D+G,\\
&F+A+B+H=C+D+E+G,\\
&D+E+F+H=A+B+C+G,
\end{aligned} \tag{6.2}$$

all lie in a $PG(2,2)$ (they are all given by linear combinations of, for example, $A+D+G+H$, $B+E+G+H$ and $F+A+B+H$). The Fano 7_3 whose vertices are the seven points (6.2) is indicated in figure 7.

A *dodecastigm* is a set of twelve general points in a projective 10-space. The homogenous coordinate sets of a dodecastigm $ABCDEFGHIJKL$ can be made to satisfy a syzygy

$$A+B+C+D+E+F+G+H+I+J+K+L=0. \tag{6.3}$$

Associated with a dodecastigm are $\binom{12}{2}=66$ *duadic* points, $\binom{12}{4}=495$ *tetradic* points and $\frac{1}{2}\binom{12}{6}=462$ *hexadic* points. In a projective space over $GF(2)$, these

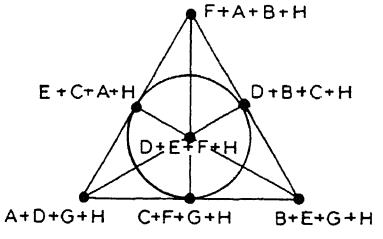


Figure 7.

associated points all lie in a 9-space, and are *all* the points of the $PG(9,2)$ in which they lie $(66+495+462=1023=2^{10}-1)$.

The set of 24 points $N_i(i\in\Omega)$ in $PG(11,2)$ constitute 2576 dodecastigms (in complementary pairs). The 63 associated points of a dodecastigm coincide with the 63 associated points of the complementry dodecastigm. The situation is analogous to that of the 12 points in $PG(5,3)$, which constitute 132 hexastigms, the associated (positive, negative and minor) points of a hexastigm coinciding with those of the complementary hexastigm.

Moreover, the 24 points constitute 759 octastigms, corresponding to the octads of a Steiner system $S(5,8,24)$. The dodecastigms correspond to the umbral dodecads of the Steiner system (Todd). The automorphism group of the Steiner system is realized as the group M_{24} of collineations generated by A, B (5.12) and F (5.16).

A set of three octastigms which together contain all 24 points corresponds to a *trio* of the Steiner system. The three octastigms of a trio have seven of their tetradic points in common—the seven points of the unique $PG(2,2)$ common to the three 6-spaces of the octastigms. The configuration of 24 points in $PG(11,2)$ can be regarded as a trio of octastigms in 3795 ways.

In order to establish the above facts, recall that the coordinatisation of $\bar{\mathscr{C}}$ can be chosen so that the $N_i(i\in\bar{Q})$ are given by the rows of the unit 12×12 matrix and the $N_i(i\in\bar{N})$ by the rows of (5.9). We thus get eleven syzygies of length 8, corresponding to eleven octastigms, each expressing the coincidence of a duadic point of the generic dodecastigm $N_i(i\in N)$ with a hexadic point of the generic dodecastigm $N_i(i\in Q)$. They are obtained by applying the powers of the transformation $B: N_i\to N_{2i}$ to the syzygy

$$N\{5\infty\}=N\{1\quad 2\quad 8\quad 16\quad 9\quad 6\}. \tag{6.4}$$

We call these *syzygies of type* (2, 6). The final row of (5.9) provides the syzygy

$$N\{0\quad 1\quad 2\quad 4\quad 8\quad 16\quad 9\quad 18\quad 13\quad 3\quad 6\quad 12\}=0 \tag{6.5}$$

of length 12 and type (0, 12), corresponding to the generic dodecastigm $N_i(i\in Q)$. (The notation NS, $S\in\Omega$ denotes $\Sigma N_i(i\in S)$). Adding these twelve syzygies in pairs gives $\binom{12}{2}$ $=132$ syzygies of type (2, 6). Each indicates an octastigm and at the same time indicates a coincidence between a duadic point of $N_i(i\in N)$ and a hexadic point of $N_i(i\in Q)$. Adding them in fours gives $\binom{12}{4}=495$ syzygies of type (4, 8). Similarly, by choosing $N_i(i\in\bar{N})$ as the reference points, we get eleven syzygies of type (6, 2), obtained from

$$N\{1\quad 0\}=N\{5\quad 20\quad 21\quad 19\quad 15\quad 14\} \tag{6.6}$$

by applying the powers of B, and one of type (12, 0), namely

$$N\{5\quad 10\quad 20\quad 17\quad 11\quad 22\quad 21\quad 19\quad 15\quad 7\quad 14\}=0. \tag{6.7}$$

Adding these twelve syzygies in twos gives 132 of type (6, 2) and adding them in fours gives 495 of type (8, 4).

495 syzygies of type (4, 4) are obtained either by adding (6.5) to those of type (4, 8) or by adding (6.7) to those of type (8, 4). We obtain the number of octastigms,

$$132+132+495=759. \tag{6.8}$$

They constitute the octads of a Steiner system $S(5,8,24)$. (The proof of this, which consists of showing that any five of the points belong to a unique octad, has been given by Todd [16]. The automorphism group of the $S(5,8,24)$ is of course the M_{24} of collineations that permutes the N_i, generated by A, B and F.

We have seen that 66 of the hexadic points of $N_i(i \in Q)$ coincide with duadic points of $N_i(i \in N)$. The remaining 396 hexadic points of $N_i(i \in Q)$ coincide with hexadic points of $N_i(i \in N)$. This provides 396 syzygies of type (6, 6). By adding (6.5) and (6.7) to these we get altogether $4 \times 396 = 1584$ of type (6, 6) (they emerge upon adding those of type (6, 2) to those of type (2, 6)). We now have all the syzygies of length 12: one of type (0, 12), one of type (12, 0), 495 each of types (4, 8) and (8, 4) and 1584 of type (6, 6). The total number of dodecastigms is

$$1 + 1 + 495 + 495 + 1584 = 2576.$$

They are the umbral dodecads of the Steiner system $S(5,8,24)$. Given a complementary pair of dodecastigms (duum of the Steiner system) their tetradic points coincide, the duadic points of one coincide with the hexadic points of the other, and the remaining hexadic points coincide. Those hexadic points of one of the dodecastigms that coincide with duadic points of the other are just the hexads of the Steiner system of the M_{12} of collineations that preserves the dodecastigm pair.

Finally, consider a generic trio of octastigms with no common point. We can take them to be the three generic octads given in (2.6), which have the syzygies

$$\begin{aligned} N\{0 \quad 8 \quad 20 \quad 14 \quad 15 \quad 3 \quad 18 \quad \infty\} &= 0, \\ N\{4 \quad 13 \quad 7 \quad 11 \quad 10 \quad 16 \quad 2 \quad 17\} &= 0, \\ N\{9 \quad 1 \quad 12 \quad 22 \quad 6 \quad 5 \quad 21 \quad 19\} &= 0. \end{aligned} \tag{6.9}$$

The $L_2(7)$ of section 2 induces the group $L_2(7)$ of collineations generated by

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
1&0&1&1&0&1&1&1&0&0&0&1\\
0&0&1&0&1&1&0&1&1&1&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0\\
0&1&1&1&0&0&0&1&0&1&1&1\\
1&0&1&1&1&0&0&0&1&0&1&1\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix},$$

$$\begin{bmatrix}
1&0&0&0&1&0&1&1&0&1&1&1\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&1&0&1&1&0&1&1&1&0&0&1\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&1&1&0&1&1&1&0&0&0&1&1\\
1&1&0&0&0&1&0&1&1&0&1&1\\
0&0&0&1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&1&0&1&1&0&1\\
1&1&0&1&1&1&0&0&0&1&0&1\\
0&0&0&1&0&1&1&0&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&0
\end{bmatrix} \tag{6.10}$$

(induced by the α, β and γ of $L_2(7)$). That they have the correct action on the N_i can be checked by multiplying I_{12} and the matrix (5.9) on the right.

The seven objects P_i(1.12) constructed from $PL(7)$ correspond, under the correspondences (2.6) to seven tetradic points that the three octastigms have in common:

$$\begin{aligned}
P_0&: N\{\ 0\ \ 8\ \ 20\ \ 15\} = N\{\ 4\ \ 13\ \ 7\ \ 16\} = N\{\ 9\ \ 1\ \ 12\ \ 6\}\\
P_1&: N\{\ 8\ \ 20\ \ 14\ \ 3\} = N\{13\ \ 7\ \ 11\ \ 16\} = N\{\ 1\ \ 12\ \ 22\ \ 5\}\\
P_2&: N\{20\ \ 14\ \ 15\ \ 18\} = N\{\ 7\ \ 11\ \ 10\ \ 2\} = N\{12\ \ 22\ \ 6\ \ 21\}\\
P_3&: N\{14\ \ 15\ \ 3\ \ 0\} = N\{11\ \ 10\ \ 16\ \ 4\} = N\{22\ \ 6\ \ 5\ \ 9\}\\
P_4&: N\{15\ \ 3\ \ 18\ \ 8\} = N\{10\ \ 16\ \ 2\ \ 13\} = N\{\ 6\ \ 5\ \ 21\ \ 1\}\\
P_5&: N\{\ 3\ \ 18\ \ 0\ \ 20\} = N\{16\ \ 2\ \ 4\ \ 7\} = N\{\ 5\ \ 21\ \ 9\ \ 12\}\\
P_6&: N\{18\ \ 0\ \ 8\ \ 14\} = N\{\ 2\ \ 4\ \ 13\ \ 11\} = N\{21\ \ 9\ \ 1\ \ 22\}.
\end{aligned} \tag{6.15}$$

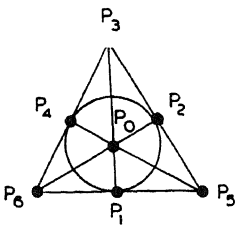


Figure 8.

The coincidences indicated by the equality signs correspond to various other octastigms. That these actually are octads of the $S(5, 8, 24)$ is readily visible in the MOG. That these seven points P_i of $PG(11, 2)$ are collinear in threes (for example $P_0 + P_1 + P_3 = 0$) is easily checked. They form a Fano configuration (figure 8) and therefore constitute a $PG(2, 2)$.

7. Twenty-four primes in $PG(11, 12)$

The words of unit weight in $\mathscr{V}$ are the sets of homogeneous coordinates for the 24 reference primes in V. We call these primes $\pi_i (i \in \Omega)$. In the canonical order (5.1) they are given by the columns (or rows) of the unit matrix I_{24}. They intersect $\bar{\mathscr{C}}$ in twelve primes of $\bar{\mathscr{C}}$, which we shall also call $\pi_i (i \in \Omega)$. From the form of the matrix N, (5.3), we see that there is just one of them, π_∞, that contains none of the 24 points $N_i (i \in \Omega)$ and that there is just one of the 24 points, N_∞ that is not contained in any of the 24 primes. The remaining 23 points and 23 primes constitute a configuration 23_{12} in $\bar{\mathscr{C}}$.

In the coordinate system on $\bar{\mathscr{C}}$ in which $N_i (i \in \bar{N})$ and $N_i (i \in \bar{Q})$ are given respectively by the rows of the matrix (5.9) and the rows of I_{12}, the primes $\pi_i (i \in \bar{N})$ and the primes $\pi_i (i \in \bar{Q})$ are given respectively by the columns of the two 12×12 matrices

$$\begin{bmatrix} Y^T & 0 \\ j^T & 1 \end{bmatrix}, \quad \begin{bmatrix} Z & j \\ j^T & 1 \end{bmatrix} \tag{7.1}$$

(i.e. by the columns of the lower half of the matrix N). The proof is analogous to the one given in §4 for the twelve primes of $\bar{W}$. Multiplying these two matrices on the left by $\Pi^{-1} = \begin{pmatrix} Q & j \\ j^T & 0 \end{pmatrix}$ gives the transpose of the matrix (5.9) and the matrix I_{12}. Hence, under the *polarity* on $\bar{\mathscr{C}}$ with matrix

$$\Pi = \begin{pmatrix} Z & j \\ j^T & 1 \end{pmatrix}, \tag{7.2}$$

the polars of the 24 points $N_i (i \in \Omega)$ are the 24 primes $\pi_i (i \in \Omega)$ (in order). (It is misleading in the context of a projective space over $GF(2)$ to speak of a 'quadric'). The configuration of 24 points and 24 primes in $PG(11, 2)$ is *self-dual*.

The group M_{24} of collineations on $\bar{V}$, which permutes the vertices of the reference simplex, is given by (5.10). Of course, it induces the group M_{24} in $\bar{\mathscr{C}}$ that permutes the 24 primes:

$$A: \pi_i \to \pi_{i\alpha}, \quad B: \pi_i \to \pi_{i\beta}, \quad H: \pi_i \to \pi_{i\gamma\delta}. \tag{7.3}$$

The M_{24} generated by A, B and H permutes the 24 primes, and leaves the point N_∞ fixed, the M_{24} generated by A, B and F permutes the 24 points and leaves π_∞ fixed, and the polarity Π interchanges the points and primes.

The two (reducible) twelve-dimensional representations of M_{24}, with α, β, $\gamma\delta$ represented, respectively, by A, B, H and by A^{-1}, B, F are adjoint to each other. That is,

$$\Pi^{-1}A\Pi = A^T, \quad \Pi^{-1}B\Pi = (B^{-1})^T, \quad \Pi^{-1}H\Pi = (F^{-1})^T. \tag{7.4}$$

Note added in proof

For a comprehensive treatment of coding theory, including properties of the Mathieu groups and other allied topics, see Conway and Sloane [18].

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Closest point of the cut locus to submanifold

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Abstract. This paper deals with some of the matching and non-matching properties of two minimal geodesics from a cut point, which is not a focal point, to the points of the submanifold. Closest cut point to the submanifold has been obtained under some additional assumptions.

Keywords. Cut-locus; closest point; submanifold; Riemannian metric; tangent bundles; Jacobi fields.

1. Introduction

Let M be a complete connected Riemannian manifold of class C^∞ and L be a compact connected imbedded submanifold of M of class C^∞. Let $N(L)$ be the normal bundle of L which is a subbundle of tangent bundle TM of M. The exponential map of the Riemannian manifold M restricted to $N(L)$ is a map $e: N(L) \to M$ of class C^∞. Let $d: M \times M \to R$ be the Riemannian distance function, then to any point $z \in M$ there is at least one point $x \in L$ such that $d(z, x) = \inf \{d(z, x') | x' \in L\}$ holds and x is said to be a point nearest to z in L. Let $z \notin L$ and consider a geodesic $c: R \to M$ of the Riemannian manifold such that $c(0) = x \in L$, $c(t) = z$ for some $t > 0$ and that the restriction of c to $[0, t]$ yields a minimal geodesic from x to z. Then the tangent vector $\dot{c}(0)$ of c is in the normal space $N_x L$ of L at x by a basic observation [2]. Since M is complete such a geodesic c always exists and consequently the map e is surjective. As to the injectivity of the map e some further concepts are essential which can be summarized as follows:

If the tangent linear map

$$T_v e: T_v N(L) \to T_{e(v)} M$$

of e at $v \in N(L)$ is not injective then v is called a focal point of L in the normal bundle $N(L)$ and $e(v)$ is said to be a focal point of L in M. The set of focal points v of L is said to be a focal locus of L in the normal bundle $N(L)$ and the set of focal points $e(v)$ of L is called the focal locus of L in M. In the special case when the submanifold L reduces to a single point $y \in M$ and consequently the normal bundle $N(L)$ coincides with the tangent space $T_y M$, the focal points of L are said to be points conjugate to y and the focal locus of L is called conjugate locus of y.

2. Cut locus of submanifold

Let $c:R\to M$ be a geodesic such that $c(0)=x\in L$, $\dot{c}(0)=w\in N_xL$, $\|w\|=1$. Since the restricted exponential map e is injective in a sufficiently small neighbourhood of the zero section in the normal bundle $N(L)$ of the submanifold L [2], consequently x is the unique nearest point of L to $z=c(t_0)$ for $0<t_0\leqslant t$ where t is sufficiently small positive value, and $c\restriction[0,t]$ is the unique minimal geodesic. Let s_w be the supremum of such values $t>0$ which is always well defined, since M is complete. If s_w is finite, then $c\restriction[0,s_w]$ is a minimal geodesic. The point $v'=(s_w w)$ is called a cut point of L in the normal bundle and $e(v')=c(s_w w)$ is called a cut point of L in M. The set of cut points of L in $N(L)$ is called the cut locus of L in $N(L)$ and the set formed by the cut points of L in M is called the cut locus of L in M.

A straightforward generalization of some basic facts established in the special case when L reduces to a single point [2] yields the following:

If $v'=s_w w\in N(L)$ is the cut point of the submanifold L then at least one of the following two assertions is true.

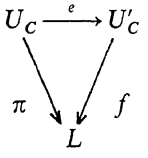
(i) The point $v'=s_w w$ is the first focal point of the submanifold L along c.
(ii) There exist at least two minimizing geodesics from two different points of the submanifold L to $e(s_w w)$.

Consider now the open neighbourhood U_C of the zero section in the normal bundle $N(L)$ given by the following definition:

$$U_C=\{tw\,|\,0\leqslant t\leqslant s_w,\ \|w\|=1,\ w\in N_xL\}.$$

In other words U_C is the neighbourhood of the zero section which is bounded by the cut locus of L in $N(L)$. (i), (ii) and some simple observations yield that the restriction of e to U_C is a diffeomorphism. Consequently $U'_C=e(U_C)$ is an open neighbourhood of the submanifold L in M.

Consider the map $f:U'_C\to L$ which maps a point $y'\in U'_C$ to the unique point $y\in L$ which is nearest to y'. Let π be the canonical projection of the normal bundle $N(L)$, then the following commutative diagram is obtained:



Consequently f is a smooth retraction of the neighbourhood U'_C to the submanifold.

3. Riemannian metric of the tangent bundles and Jacobi fields

Let M be an n-dimensional complete connected Riemannian manifold with metric g. Then a metric $\bar{g}$ of its tangent bundle TM is given by [1]

$$\bar{g}(Z,Z)=g(Tp_M(Z),Tp_M(Z))+g(K(Z),K(Z))$$

for any $Z \in T\dot{T}M$, where $Tp_M : TTM \to TM$ is the tangent linear map corresponding to the canonical projection $p_M : TM \to M$ and $K : TTM \to TM$ is the connector map. It is well known that for every $v \in TM$ the fibre $T_v TM$ of the canonical projection $p_{TM} : TTM \to TM$ has a unique orthogonal decomposition

$$T_v TM \to H_v TM \oplus V_v TM,$$

where the subspaces

$$H_v TM = \mathrm{Ker}(K \restriction T_v TM),$$
$$V_v TM \mathrm{Ker}(Tp_M \restriction T_v TM)$$

are called the horizontal and vertical subspaces of $T_v TM$ respectively. Moreover the following conditions are satisfied:

(a) Both of the canonical isomorphisms

$$h_v^{-1} = Tp_M \restriction H_v TM : H_v TM \to T_x M$$

and $i_v = K \restriction V_v TM : V_v TM \to T_x M$ are isometries for any $v \in TM$, where $x = p_M(v)$.
(b) The canonical projection $p_M : TM \to M$ is a Riemannian submersion.

Let $v : V \to TM$ be a vector field on a neighbourhood V of $x \in M$, then for the covariant derivative of v in the direction $v_1 \in T_x M$ we have $v_2 = D_{v_1} v(x) = K \cdot Tv(v_1)$ where Tv is the tangent linear map defined by v. Since $Tv(v_1) \in T_v TM$

$$Tv(v_1) = h_v(v_1) + i_v^{-1}(v_2)$$

holds by the above decomposition. More generally for any $Z \in TTM$, $Z = h_v(v_1) + i_v^{-1}(v_2)$ holds, where $v_1 = Tp_M(Z), v_2 = K(Z)$ and $v = p_{TM}(Z)$. Also any $Z \in TTM$, $p_{TM}(Z) \neq 0$, defines uniquely a Jacobi field

$$J : [0, \infty) \to TM$$

along the geodesic $c : [0, \infty) \to M$ such that $\dot{c}(0) = v = p_{TM}(Z)$, and the following initial conditions are considered.

$$J(0) = v_1 = Tp_M(Z),$$
$$\dot{J}(0) = v_2 = K(Z).$$

Now let L be an m-dimensional compact connected submanifold of M where $m < n$. Let $N(L)$ be the normal bundle of L. Let $v : V \to N(L)$ be a section of the normal bundle $N(L)$ on a neighbourhood V of $x \in L$, then for the covariant derivative of v in the direction $v_1 \in T_x L$ we have

$$v_2 = D_{v_1} v(x) = K \cdot Tv(v_1)$$

holds as before. But considering the orthogonal decomposition $T_x M = T_x L \oplus N_x L$ of the tangent space $T_x M$, $v_2 = A_v(v_1) + \tilde{v}_2$ is valid where $A_v : T_x L \to T_x L$ is the Weingarten map corresponding to the normal $v(x) \in N_x L$ and $\tilde{v}_2 = \tilde{D}_{v_1} v(x) \in N_x L$. The inclusions $N(L) \subset TM$ and $TN(L) \subset TTM$ permit to consider an induced

connector map $\tilde{K}: TN(L) \to N(L)$ given by $\tilde{K}(Z) = K(Z) - A_v(v_1)$ for any $Z \in TN(L)$ where $v = p_{TM} \restriction TN(L)(Z) \in N_x L$ and $v_1 = Tp_M \restriction TN(L)(Z) \in T_x L$. Thus the induced covariant derivative of v in the direction v_1 is given by $\tilde{v}_2 = \tilde{D}_{v_1} v(x) = \tilde{K}\, Tv(v_1) \in N_x L$. Thus the canonical metric $\tilde{g}$ of the normal bundle $N(L)$ can be given as

$$\tilde{g}(Z, Z) = g(Tp_M(Z), Tp_M(Z) + g(\tilde{K}(Z), \tilde{K}(Z)))$$

for any $Z \in TN(L)$.

It is easy to see that for every $v \in N(L)$ the fibre $T_v N(L)$ of the canonical projection

$$p_{TM} \restriction TN(L) = TN(L) \to N(L)$$

has a unique orthogonal decomposition

$$T_v N(L) = \tilde{H}_v N(L) \oplus \tilde{V}_v N(L), \quad \text{where the subspaces}$$
$$\tilde{H}_v N(L) = \mathrm{Ker}(\tilde{K} \restriction T_v N(L)),$$
$$\tilde{V}_v N(L) = \mathrm{Ker}(Tp_M \restriction T_v N(L))$$

are called the horizontal and vertical subspaces of $T_v NL$ respectively. Moreover the following conditions are satisfied.

(i) Both of the isomorphisms

$$\tilde{h}_v^{-1} = Tp_M \restriction \tilde{H}_v N(L) : \tilde{H}_v N(L) \to T_x : L$$

and $\tilde{i}_v = \tilde{K} \restriction \tilde{V}_v N(L) : \tilde{V}_v N(L) \to N_x L$ are isometries for any $v \in N(L)$, where $x = p_M(v) \in L$;
(ii) The canonical projection

$$p_M \restriction N(L) : N(L) \to L$$

is a Riemannian submersion.

It may be noted that for any $Z \in TN(L) \subset TTM$ we have already two orthogonal decompositions. In the first case the components of Z, $Z_{\mathrm{hor}} = h_v(v_1)$ and

$$Z_{\mathrm{vert}} = i_v^{-1}(v_2)$$

are orthogonal in the metric g of TM. In the second case Z has the following components

$$\tilde{Z}_{\mathrm{hor}} = Z_{\mathrm{hor}} + i_v^{-1}(A_v(v_1))$$

and

$$\tilde{Z}_{\mathrm{vert}} = Z_{\mathrm{vert}} - i_v^{-1}(A_v(v_1)),$$

which are evidently orthogonal with respect to the metric $\tilde{g}$ of the normal bundle $N(L)$.

Let $\Psi: TN(L) \to TM$ be the map defined by $\Psi(Z) = v_1 + v_2 - A_v(v_1) \in TM$ where $Z \in TN(L)$, $x = p_M(v)$. Then it is easy to see that for any $v \in N(L)$ the map $\Psi \restriction T_v N(L)$: $T_v N(L) \to T_x M$ is an isometry if the metrics $\tilde{g}$ and g are considered. Now a characterization of the so-called L-Jacobi fields will be given below.

It is clear that if $v \in N(L)$ is non-zero and $\rho: [0, \infty) \to N(L)$ is a ray of $N(L)$ defined by $\rho(t) = t \cdot v$, for $t \in [0, \infty)$, then $\dot{\rho}(1) \in T_v N(L)$ holds. Let $Z \in TN(L)$ such that

$v = p_{TM}(Z) \neq 0$ and $\tilde{g}(\dot{\rho}(1), Z) = 0$ holds. Then Z defines uniquely an L-Jacobi field [2] $J:[0, \infty) \to TM$ along the geodesic $c:[0, \infty) \to M$ such that $\dot{c}(0) = v = p_{TM}(Z) \in N_x L$, and the following initial conditions are considered:

$$J(0) = v_1 = Tp_M(Z) \in T_x L,$$
$$J'(0) = v_2 = K(Z).$$

Notice that the conditions $g(v, v_1) = 0$ and $g(v, v_2) = \tilde{g}(\dot{\rho}(1), Z) = 0$ imply that $g(\dot{c}, J) = 0$ is valid for every parameter along the geodesic c, on the other hand

$$\tilde{K}(Z) = v_2 - A_v(v_1) \in N_x L \quad \text{holds as well.}$$

Lemma 3.1. Let L be a smooth connected submanifold of a complete connected Riemannian manifold M. Let $e:N(L) \to M$ be the restricted exponential map. If $v \in N(L)$ is a non-zero vector and ρ is the ray of $N(L)$ defined by v, then

$$\tilde{g}(\dot{\rho}(1), Z) = g(T_v e(\dot{\rho}(1)), T_v e(Z))$$

holds for any $Z \in T_v N(L)$, where $T_v e$ denotes the tangent linear map of e at v.

Proof. Let $Z \in T_v N(L)$. Then Z can be represented by a smooth variation α of the geodesic $c = e(\rho)$:

$$\alpha(s, t) = e(s, v(t)), \quad \text{for} \quad s, t \in R \quad \text{where} \quad Z = \left.\frac{\mathrm{d}\alpha(s, t)}{\mathrm{d}s}\right|_{s=0}.$$

In other words, $Z \in T_v N(L)$ defines a Jacobi field J along the geodesic c such that $\dot{c}(0) = v = p_{TM}(Z)$

$$v_1 = Tp_M(Z)$$
$$J' = v_2 = K(Z)$$

are the initial conditions and

$$J(1) = T_v e(Z), \quad \dot{c}(1) = T_v e(\dot{\rho}(1))$$

are also valid. The Jacobi equation $J'' + R(J, \dot{c})\dot{c} = 0$ implies that $g(\dot{c}, J'') = 0$ and so $g(\dot{c}, J) = c_1 t + c_2$ holds. Applying the definition of the metric $\tilde{g}$

$$\tilde{g}(\dot{\rho}(1), Z) = g(0, v_1) + g(v, v_2 - A_v(v_1)) = g(\dot{c}(0), J'(0))$$

is valid after a short computation. Substituting $t = 1$ and

$$c_1 = g(\dot{c}(0), J'(0)) = \tilde{g}(\dot{\rho}(1), Z),$$
$$c_2 = g(\dot{c}(0), J(0)) = 0,$$

we get the desired result

$$g(\dot{c}(1), J(1)) = \tilde{g}(\dot{\rho}(1), Z).$$

COROLLARY

Let L be a connected submanifold of a complete connected Riemannian manifold M. Let $e:N(L)\to M$ be the restricted exponential map. If $v\in N(L)$ is a non-zero vector then the kernel of the map $T_v e$ is orthogonal to $\dot{\rho}(1)$ where ρ is the ray defined by v.

Proof. Let $Z\in \mathrm{Ker}\, T_v e$, then $g(T_v e(\dot{c}(1), T_v e(Z))) = 0$. Using the above lemma we get the result.

4. Closest point of the cut locus to submanifold

Let $c:[0,a]\to M$ be a normal geodesic such that $c(0)=z\in L, \dot{c}(0)\in N_z L$. Let x be a cut point of L along c. If c' is another geodesic from the cut point x to a different point $z_1\in L$ and orthogonal to L such that c' does not match smoothly with c at x, then x is not the closest cut point of L. This fact is stated in the following theorem.

Theorem 4.1. *Let M be a complete connected Riemannian manifold of class C^∞ and L be a compact connected submanifold of M of class C^∞. Let $e(v)=x\in M$, $v\in N(L)$ be a point of the cut locus $C(L)$ to L which is not a focal point of L. If c_1 and c_2 are two different minimizing geodesics from x to L such that they do not match smoothly at x. Then there is a cut point of L in any neighbourhood of x which is nearer to L than x.*

We have used the following lemma [8] in proving the theorem.

Lemma. Let M be a complete connected Riemannian manifold and L be a compact connected submanifold of M. Let $c:[0,a]\to M$ be a minimal geodesic from $c(a)$ to L. If c' is the part of c then c' minimizes the distance uniquely from its end point $c'(b)$ to the points of L for any value of the parameter $b<a$.

Proof of the theorem. For the sake of convenience the zero-section of the normal bundle $N(L)$ is identified with the submanifold L. Let $p_L:N(L)\to L$ be the projection map. Since $N(L)$ is a vector bundle, it is locally trivial, that is, every point y of the zero-section of $N(L)$ has a neighbourhood U such that $p_L^{-1}(U)$ is isomorphic with $U\times N_y L$. There is a Riemannian metric on $N(L)$ by basic results studied in § 3 and with respect to this metric $T_v e$ preserves the length of vectors tangent to the rays of $N(L)$ by lemma (3.1). Let $v\in N_y L$ be a non-zero vector where $y\in U$, then the locus of the end points of such v with fixed length will be the sphere of dimension $n-m-1$. Consider with v the family of vectors of the same length as v in $p_L^{-1}(U)$ then corresponding to these vectors there is a union of the spheres which forms a piece of a hypersurface say K and the tangent space $T_v K$ at v is orthogonal to v which is proved as follows:

Let $\rho:[0,\infty)\to N(L)$ be a ray of $N(L)$ defined by $\rho(t)=t\cdot v$, $t\in[0,\infty)$ then $\dot{\rho}(1)=v$. Consider a geodesic variation $w_t(s), t\in[0,1]$, $s\in(-\varepsilon,\varepsilon)$ of v considered as a geodesic in $N(L)$ such that $w_t=(0)=v$ and varied geodesics are of the same lengths and orthogonal to U. Then the variation vector field

$$W=\left.\frac{\partial w_t(0)}{\partial s}\right|_{t=1}$$

will be a Jacobi field tangent to the hypersurface K at v and is a non-zero field at v, since v is not a focal point of L in $N(L)$. This Jacobi field will be orthogonal to v at $t=1$. This process can be applied for any element of T_vK. Since each variation vector field which is an element of T_vK is orthogonal to v therefore T_vK will be orthogonal to v, i.e. for $X \in T_vK, \tilde{g}(\dot{\rho}(1), X) = 0$ holds, where $\tilde{g}$ is the induced metric of $N(L)$. Now we define geodesic $c_1 : [0,1] \to M$ such that $c_1(0) = z_1 \in L, \dot{c}_1(0) \in N_{z_1}L, c_1(1) = x = e(v)$. Consider for c_1 a family of neighbouring geodesics each orthogonal to L then under the restricted exponential map e each member of this family is the image of non-zero vectors taken in $p_L^{-1}(U)$ corresponding to v. Since each vector in $p_L^{-1}(U)$ is of the same length as that of v, their images under the map e will be of the same lengths too. Since $e(v) = x$ is not a focal point of L in M, the image $e(K)$ will be a piece of hypersurface containing x in M. Since T_vK is orthogonal to $\dot{\rho}(1)$, therefore by lemma (3.1), $T_v e(T_vK) = T_{e(v)}e(K)$ will be orthogonal to $\dot{c}(1)$, i.e. for $Z = T_v e(x) \in T_{e(v)}e(K)$,

$$\dot{c}_1(1) = T_v e(\dot{\rho}(1)),$$

$$g(\dot{c}_1(1), Z) = g(T_v e(\dot{\rho}(1)), T_v e(X))$$

$$= \tilde{g}(\dot{\rho}(1), X) = 0,$$

where g is the Riemannian metric of M. Consequently the hypersurface $e(K)$ is orthogonal to c_1. Similar result holds for the geodesic c_2 passing orthogonally through the point $z_2 \in L$ to x. Since c_1 and c_2 do not match smoothly at x, the two tangent hyperplanes at x do intersect in the neighbourhood of x. From this there exists a point x' close to x and geodesics c_1' and c_2' neighbouring to c_1 and c_2 and shorter than them respectively such that c_i' join to L. But this is adequate to conclude that there is a cut point x'' to L closer to L than x. In fact if the length of either c_1' or c_2' is greater than the distance of x' to L, there must be a cut point on that geodesic even before it reaches x'. If length $c_1' =$ length $c_2' =$ distance of x' to L, then x' is a cut point. Now there are two possibilities:

(i) If $z_1 \neq z_2$, then the minimizing geodesics from x' have different foot points. Consequently they are different. But then x' is a cut point of L nearer to L than x.

(ii) If $z_1 = z_2$, then since c_1 and c_2 are minimizing geodesics and not matching smoothly at x, there will be two different subspaces of codimension 1 in T_xM each orthogonal to c_1 and c_2 respectively at x and both the subspaces intersect in the neighbourhood of x. Therefore there is a point x' in the neighbourhood of x such that there are two different minimizing geodesics from x' to L and x' is nearer to L than x. Consequently x' is a cut point of L nearer to L than x.

Theorem 4.2. *Let M be a complete connected Riemannian manifold and L be a compact connected submanifold of M. Let the cut locus of L be non-empty and let $e(v)$ be a closest point of the cut locus to L. If $e(v)$ is not a focal point of L. Then there are at most two different points of L which are at minimal distance from $e(v)$.*

Proof. Let $e(v)$ be not a focal point of L, then there are at least two different minimizing geodesics from L to the cut point $e(v)$. We have to show that there are at most two different points of L which are foot points of minimizing geodesics from L to $e(v)$. Let c_1 and c_2 be any two minimal geodesics through $e(v)$ to two different points z_1

and z_2 of L. If c_1 and c_2 do not match smoothly at $e(v)$, then there is a point x' which is nearer to L than $e(v)$. But then by theorem (4.1) the point x' will be a cut point of L, which contradicts the fact that $e(v)$ is the closest cut point of the submanifold L. Thus if c_1 and c_2 are two minimizing geodesics from two different points of L to $e(v)$ and $e(v)$ is the closest cut point of L then they match smoothly at $e(v)$. It remains to prove that there are at most two different points of L which are at minimal distance from $e(v)$. To prove this we assume that z_3 is a third foot point of a minimal geodesic passing through $e(v)$. But then neither minimizing geodesics c_1, c match nor c, c_2 match at $e(v)$ in consequence of the fact that c_1 and c_2 match smoothly at $e(v)$. Hence assertion of the theorem follows.

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t-structures in the derived category of representations of quivers

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Abstract. Given a finite quiver without oriented cycles, we describe a family of algebras whose module category has the same derived category as that of the quiver algebra. This is done in the more general setting of t-structures in triangulated categories. A completeness result is shown for Dynkin quivers, thus reproving a result of Happel [H].

Keywords. Quivers; module category of algebras; t-structures; tilting; realization functor.

Introduction

The notion of a t-structure in a triangulated category is introduced in [1, 1–3]. Given a t-structure in a full thick subcategory D_U of D and a t-structure in the quotient, under some hypotheses, there is a notion of glueing (= 'recollement' [1, 1–4]) which produces a t-structure in D. We apply these constructions to the (bounded) derived category $D^b(\mathrm{mod}\,(\Lambda, \Omega))$ of the category $\mathrm{mod}\,(\Lambda, \Omega)$ of representations of a quiver (Λ, Ω). In § 2 we define 'data' in (Λ, Ω) and attach them to t-structures in $D^b(\mathrm{mod}\,(\Lambda, \Omega))$. Our main result, Theorem 3.3, gives a necessary and sufficient condition for the heart of this t-structure to give back the derived category $D^b(\mathrm{mod}\,(\Lambda, \Omega))$. This condition is easy to verify in practice. Some examples are given in § 5. When this condition is satisfied, Theorem 7.1 asserts that the heart of the t-structure can be identified to the category $\mathrm{mod}\,(B)$ where B is an algebra obtained as an n-step ($n = \#\Lambda$) tilting of the quiver algebra of $(\Lambda, \bar{\Omega})$ where $(\Lambda, \bar{\Omega})$ is another quiver obtained from (Λ, Ω) by changing the direction of some arrows.

In § 4 we prove that the intermediate t-structures that arise from disregarding tails of 'data' have the property that the realization functor (§ 3) is an equivalence if the final t-structure is so. This allows us to use the inductive arguments of § 6 leading to Theorem 7.1. The fact that if (Λ, Ω) is a Dynkin quiver then it is of finite representation type can be used to deduce from Theorem 7.1 a completeness result for this case (Theorem 8.5). This is related to a result of Happel [5, § 5].

1. Reflection functors R_α^+, R_α^-

Let k be a field assumed algebraically closed. Let (Λ, Ω) be a finite quiver (:= a finite set of vertices linked with arrows) without oriented cycles. Here Λ denotes the

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underlying set of vertices and Ω the orientation data for the edges of the graph. A representation V of (Λ, Ω) assigns a vector space V_α (over k) to each vertix $\alpha \in \Lambda$ and a linear map $V_{\overrightarrow{\alpha\beta}}: V_\alpha \to V_\beta$ to each arrow $\alpha \to \beta$. A morphism between two such representations V and W is a collection of linear maps $f(\alpha): V_\alpha \to W_\alpha$, $(\alpha \in \Lambda)$ satisfying $f(\beta) \cdot V_{\overrightarrow{\alpha\beta}} = W_{\overrightarrow{\alpha\beta}} \cdot f(\alpha)$ for each arrow $\alpha \to \beta$ of (Λ, Ω). Denote by $\mathrm{mod}(\Lambda, \Omega)$ the (abelian) category of finite dimensional representations of (Λ, Ω) $(:= \sum_\alpha \dim V(\alpha) < \infty)$.

(1.1) The category $\mathrm{mod}(\Lambda, \Omega)$ can be identified with the category of finitely generated (left) modules over a finite dimensional k-algebra $k[\Lambda, \Omega]$ called the quiver algebra of (Λ, Ω) (cf. [4], § 4).

If $\alpha \in \Lambda$, α is said to be a sink (resp. source) if there are no arrows starting at α (resp. ending at α). If $\alpha \in \Lambda$, $(\Lambda, s_\alpha \Omega)$ is a new quiver obtained from (Λ, Ω) by reversing the arrows at α.

(1.2) The orientation Ω in (Λ, Ω) is called admissible if there exists an enumeration $\alpha_1, \ldots, \alpha_n$ of Λ such that α_1 is a sink of (Λ, Ω), α_2 is a sink of $(\Lambda, s_{\alpha_1}\Omega), \ldots, \alpha_i$ is a sink of $(\Lambda, s_{\alpha_{i-1}} \cdots s_{\alpha_2} \cdot s_{\alpha_1} \Omega)$ $(1 \leqslant i \leqslant n)$. In this case $\alpha_1, \ldots, \alpha_n$ is called an admissible enumeration. All the quivers that we encounter in this article shall have an admissible orientation.

For an introduction to triangulated categories and general notions associated with them and their study we refer to [7] and [6]. Derived categories and derived functors which we use here are as in [7] and [6]. Sometimes, we say triangles instead of distinguished triangles.

If α is a sink or a source one has 'reflection' functors $\mathrm{mod}(\Lambda, \Omega) \to \mathrm{mod}(\Lambda, s_\alpha \Omega)$ which were introduced by Gelfand, Bernstein and Ponomarev. We do not need the exact definitions here which can be found in [3, pp. 15, 16]. However, we state here in a convenient form the important fact that the derived functors of these reflection functors become isomorphisms between derived categories. This fact which was proved by Happel [5] in a more general context (namely, tiltings) was also independently proved by the author (unpublished). The form in which we state it here is closer to the latter version.

(1.3) It is convenient to denote an object V of $\mathrm{mod}(\Lambda, \Omega)$ by $\{V_\alpha, \ldots, V_{\overrightarrow{\alpha\beta}}, \ldots\}$ (see definition before 1.1). A complex $V^\cdot, d)$ (where d is the differential) of objects of $\mathrm{mod}(\Lambda, \Omega)$ will be denoted by $\{V^\cdot_\alpha, \ldots, V^\cdot_{\overrightarrow{\alpha\beta}}, \ldots\}$. Let E be the set of arrows of (Λ, Ω) whose end point is α. (Here, α is supposed to be a sink of (Λ, Ω).) Then we have a homomorphism

$$\bigoplus_{\overrightarrow{\beta\alpha} \in E} V^\cdot_\beta \to V^\cdot_\alpha \tag{1.3a}$$

by summing the $V^\cdot_{\overrightarrow{\beta\alpha}}$ for $\overrightarrow{\beta\alpha} \in E$. Note that 1.3a is a morphism of complexes of vector spaces. Let $M^\cdot$ be the mapping cone on the negative of the morphism 1.3a. By definition of the mapping cone construction, we have canonical morphism $M^\cdot[-1] \to$ left side of 1.3a $\{M^\cdot \to M^\cdot[1]$ is the translation functor$\}$. Now we define $V'^\cdot_\alpha = M^\cdot[-1]$ and $V'^\cdot_\gamma = V^\cdot_\gamma$ if $\gamma \neq \alpha$ $(\gamma \in \Lambda)$. Thus, we have a canonical morphism

$$V'^\cdot_\alpha \to \bigoplus_{\overrightarrow{\beta\alpha} \in E} V^\cdot_\beta,$$

which, by projecting into summands induces morphisms

$$V'^\cdot_\alpha \to V^\cdot_\beta \quad (\text{for } \overrightarrow{\beta\alpha} \in E).$$

We denote their negatives by $V^{\cdot}_{\overrightarrow{\alpha\beta}}: V^{\cdot}_{\alpha} \to V^{\cdot}_{\beta} (= V'^{\cdot}_{\beta})$. (Our sign conventions coincide with those in [V] but differ from [6].) Also, we define $V^{\cdot}_{\overrightarrow{\mu\nu}}: V^{\cdot}_{\mu} \to V^{\cdot}_{\nu}$ to be equal to the given $V^{\cdot}_{\overrightarrow{\mu\nu}}$ if μ and ν are different from α. $\{V^{\cdot}_{\mu}, \ldots, V^{\cdot}_{\overrightarrow{\mu\nu}}, \ldots\}$ is a complex of objects of $\mathrm{mod}(\Lambda, s_\alpha\Omega)$. We define (1.3b) $R^+_\alpha\{V^{\cdot}_{\mu}, \ldots, V^{\cdot}_{\overrightarrow{\mu\nu}}, \ldots\} = \{V'^{\cdot}_{\mu}, \ldots, V'^{\cdot}_{\overrightarrow{\mu\nu}}, \ldots\} R^+_\alpha$ is functorial for morphisms of chain complexes. It takes homotopy equivalent morphisms to homotopy equivalent morphisms and quasi-isomorphisms to quasi-isomorphisms. Thus, R^+_α gives rise to a functor between the derived categories of $\mathrm{mod}(\Lambda, \Omega)$ and $\mathrm{mod}(\Lambda, s_\alpha\Omega)$.

By the result [5, Theorem 1.6] of Happel.

PROPOSITION 1.4

Let $D^b(\mathrm{mod}(\Lambda, \Omega))$ {*resp.* $D^b(\mathrm{mod}(\Lambda, s_\alpha\Omega))$} *denote the bounded derived category of* $\mathrm{mod}(\Lambda, \Omega)$ {*resp.* $\mathrm{mod}(\Lambda, s_\alpha\Omega)$}. *Then* $R^+_\alpha: D^b(\mathrm{mod}(\Lambda, \Omega)) \to D^b(\mathrm{mod}(\Lambda, s_\alpha\Omega))$ *is an equivalence of triangulated categories.*

(1.5) Dually, if α is a source, there is an analogous result. Let E be the set of arrows of (Λ, Ω) whose initial point is α. Then we have a homomorphism

$$V^{\cdot}_{\alpha} \to \bigoplus_{\overrightarrow{\alpha\beta} \in E} V^{\cdot}_{\beta}, \tag{1.5a}$$

where $V^{\cdot}$ is a complex of objects of $\mathrm{mod}(\Lambda, \Omega)$ as in 1.3. Let $M^{\cdot}$ be the cone on the negative of 1.5a. We now define $V'^{\cdot}_{\alpha} = M^{\cdot}$ and $V'^{\cdot}_{\mu} = V^{\cdot}_{\mu}$ if $\mu \neq \alpha (\mu \in \Lambda)$. The cone construction yields a canonical morphism $\bigoplus_{\overrightarrow{\alpha\beta} \in E} V^{\cdot}_{\beta} \to V'^{\cdot}_{\alpha}$, which by restriction to the summands induces morphisms

$$V'^{\cdot}_{\overrightarrow{\beta\alpha}}: V'^{\cdot}_{\beta} \to V'^{\cdot}_{\alpha} \quad (\text{for } \overrightarrow{\alpha\beta} \in E). \tag{1.5b}$$

If μ and ν are different from α, we define

$$V'^{\cdot}_{\overrightarrow{\mu\nu}}: V'^{\cdot}_{\mu} \to V'^{\cdot}_{\nu}$$

to be equal to the given $V^{\cdot}_{\overrightarrow{\mu\nu}}$ in $V^{\cdot}$.

We define

$$R^-_\alpha\{V^{\cdot}_{\mu}, \ldots, V^{\cdot}_{\overrightarrow{\mu\nu}}, \ldots\} = \{V'^{\cdot}_{\mu}, \ldots, V'^{\cdot}_{\overrightarrow{\mu\nu}}, \ldots\}. \tag{1.5c}$$

Similar to 1.4 we have

PROPOSITION 1.6

$R^-_\alpha: D^b(\mathrm{mod}(\Lambda, \Omega)) \to D^b(\mathrm{mod}(\Lambda, s_\alpha\Omega))$ *is an equivalence of triangulated categories.*

Remark 1.7 One can show that $R^-_\alpha \circ R^+_\alpha$ and $R^+_\alpha \circ R^-_\alpha$ are naturally equivalent to the identity functor.

(1.8) Let α be a sink of (Λ, Ω). Let (Λ_U, Ω_U) be the quiver obtained by deleting α from (Λ, Ω). We can regard an object of $\mathrm{mod}(\Lambda_U, \Omega_U)$ as an object of $\mathrm{mod}(\Lambda, \Omega)$ {resp. $\mathrm{mod}(\Lambda, s_\alpha\Omega)$} by extending by zero over α. This defines obvious functors $j_*: \mathrm{mod}(\Lambda_U, \Omega_U) \to \mathrm{mod}(\Lambda, \Omega)$ and $\tilde{j}_*: \mathrm{mod}(\Lambda_U, \Omega_U) \to \mathrm{mod}(\Lambda, s_\alpha\Omega)$ which are exact.

We denote the derived functors by the same symbols. Thus we have defined

$$j_*: D^b(\mathrm{mod}\,(\Lambda_U, \Omega_U)) \to D^b(\mathrm{mod}\,(\Lambda, \Omega)), \tag{1.8a}$$

and

$$\tilde{j}_*: D^b(\mathrm{mod}\,(\Lambda_U, \Omega_U)) \to D^b(\mathrm{mod}\,(\Lambda, s_\alpha\Omega)). \tag{1.8b}$$

We define

$$j_!: D^b(\mathrm{mod}\,(\Lambda_U, \Omega_U)) \to D^b(\mathrm{mod}\,(\Lambda, \Omega)), \tag{1.8c}$$

by $j_! = R_\alpha^- \circ \tilde{j}_*$ where $R_\alpha^-: D^b(\mathrm{mod}\,(\Lambda, s_\alpha\Omega)) \to D^b(\mathrm{mod}\,(\Lambda, \Omega))$ is the isomorphism 1.5c. An object of $\mathrm{mod}\,(\Lambda, \Omega)$ gives rise to an object of $\mathrm{mod}\,(\Lambda_U, \Omega_U)$ by restriction. This gives rise to an exact functor $j^*: \mathrm{mod}\,(\Lambda, \Omega) \to \mathrm{mod}\,(\Lambda_U, \Omega_U)$ and its derived functor

$$j^*: D^b(\mathrm{mod}\,(\Lambda, \Omega)) \to D^b(\mathrm{mod}\,(\Lambda_U, \Omega_U)) \tag{1.9}$$

Lemma 1.10

(i) *$j_!$ is left adjoint to j^**
(ii) *j_* is right adjoint to j^*.*

This lemma will be proved along with Lemma 1.15 below.

We can regard a k-vector space V as an object of $\mathrm{mod}\,(\Lambda, \Omega)$ by setting $V_\beta = 0$ for $\beta \neq \alpha (\beta \in \Lambda)$, $V_\alpha = V$ and $V_{\overrightarrow{\beta\alpha}} = 0$ for all arrows. This defines an obvious exact functor $i_*: \mathrm{mod}\, k \to \mathrm{mod}\,(\Lambda, \Omega)$ and its derived functor

$$i_*: D^b(\mathrm{mod}\, k) \to D^b(\mathrm{mod}\,(\Lambda, \Omega)). \tag{1.11}$$

Since an object V of $\mathrm{mod}\,(\Lambda, \Omega)$ {resp. $\mathrm{mod}\,(\Lambda, s_\alpha\Omega)$} assigns a vector space V_β for each $\beta \in \Lambda$, we have an exact functor $i^!: \mathrm{mod}\,(\Lambda, \Omega) \to \mathrm{mod}\, k$ {resp. $\tilde{i}^!: \mathrm{mod}\,(\Lambda, s_\alpha\Omega) \to \mathrm{mod}\, k$} defined by $V \mapsto V_\alpha$ and derived functors

$$i^!: D^b(\mathrm{mod}\,(\Lambda, \Omega)) \to D^b(\mathrm{mod}\, k) \tag{1.12}$$

and

$$\tilde{i}^!: D^b(\mathrm{mod}\,(\Lambda, s_\alpha\Omega)) \to D^b(\mathrm{mod}\, k). \tag{1.13}$$

We define

$$i^*: D^b(\mathrm{mod}\,(\Lambda, \Omega)) \to D^b(\mathrm{mod}\, k) \tag{1.14}$$

by $i^*(X) = (\tilde{i}^! \circ R_\alpha^+)(X[1])$

Lemma 1.15

(i) *i^* is left adjoint to i_**
(ii) *$i^!$ is right adjoint to i_*.*

Before proving 1.10 and 1.15, we observe the easy relations

$$i^* \circ i_* = \text{identity} = i^! \circ i_* \tag{1.15a}$$

and

$$j^* \circ j_* = \text{identity} = j^* \circ j_!. \tag{1.15b}$$

Proof of 1.10 and 1.15. Let $M \in D^b(\mathrm{mod}(\Lambda, \Omega))$ and $N_U \in D^b(\mathrm{mod}(\Lambda_U, \Omega_U))$. Then one sees from definitions that $\mathrm{Hom}(j^*M, N_U) \approx \mathrm{Hom}(M, j_* N_U)$ which implies 1.10 (ii). Also,

$$\begin{aligned}\mathrm{Hom}(N_U, j^*M) &= \mathrm{Hom}(N_U, \tilde{j}^* R_\alpha^+ M)\\ &= \mathrm{Hom}(\tilde{j}_* N_U, R_\alpha^+ M)\\ &= \mathrm{Hom}(R_\alpha^- \tilde{j}_* N_U, M) \qquad \text{(by 1.7)}\\ &= \mathrm{Hom}(j_! N_U, M) \qquad \text{(by 1.8c)}\end{aligned}$$

which proves 1.10 (i).

Let $M \in D^b(\mathrm{mod}(\Lambda, \Omega))$ and $V \in D^b(\mathrm{mod}\, k)$. Then it is trivial to see that $\mathrm{Hom}(i_* V, M) = \mathrm{Hom}(V, i^! M)$. Also,

$$\begin{aligned}\mathrm{Hom}(M, i_* V) &= \mathrm{Hom}(R_\alpha^+ M, R_\alpha^+ i_* V) \quad (1.4)\\ &= \mathrm{Hom}(R_\alpha^+ M, \tilde{i}_*(V[-1]))\\ &= \mathrm{Hom}(\tilde{i}^! R_\alpha^+ M, V[-1])\\ &= \mathrm{Hom}(i^* M, V). \qquad (1.14).\end{aligned}$$

This proves 1.15. (q.e.d.)

(1.16) The functors i^*, i_*, $i^!$, $j_!$, j^*, j_* satisfy the formalism of [1, § 1.4.3] namely the properties 1.4.3.1 thru 1.4.3.5 as can be checked easily using the definitions. That is all that one needs to apply 'recollement' [1, § 1.4].

(1.16a) We recall the definition of a '*t*-structure' [1, § 1.3]. It is a data $(D^{\leq 0}, D^{\geq 0})$ where $D^{\leq 0}$ and $D^{\geq 0}$ are full subcategories of $D^b(\mathrm{mod}(\Lambda, \Omega))$ satisfying axioms (i), (ii) and (iii) below. For any integer n write $D^{\leq n} = D^{\leq 0}[-n]$ and $D^{\geq n} = D^{\geq 0}[-n]$

(i) $\mathrm{Hom}(X, Y) = 0$ for $X \in D^{\leq 0}$ and $Y \in D^{\geq 1}$.
(ii) $D^{\leq 0} \subseteq D^{\leq 1}$ and $D^{\geq 0} \supseteq D^{\geq 1}$.
(iii) For any $X \in D^b(\mathrm{mod}(\Lambda, \Omega))$ there exists a distinguished triangle $A \to X \to B$ such that $A \in D^{\leq 0}$ and $B \in D^{\geq 1}$.

PROPOSITION 1.16b

Let $(D_U^{\leq 0}, D_U^{\geq 0})$ be a t-structure in $D^b(\mathrm{mod}(\Lambda_U, \Omega_U))$ and $(D_F^{\leq 0}, D_F^{\geq 0})$ a t-structure in $D^b(\mathrm{mod}\, k)$. Define

$$\tilde{D}^{\leq 0} = \{K \in D^b(\mathrm{mod}(\Lambda, \Omega)) \mid j^* K \in D_U^{\leq 0} \text{ and } i^* K \in D_F^{\leq 0}\}$$

and

$$\tilde{D}^{\geq 0} = \{K \in D^b(\mathrm{mod}(\Lambda, \Omega)) \mid j^* K \in D_U^{\geq 0} \text{ and } i^! K \in D_F^{\geq 0}\}.$$

Then $(\tilde{D}^{\leq 0}, \tilde{D}^{\geq 0})$ is a t-structure in $D^b(\mathrm{mod}(\Lambda, \Omega))$.

The proposition is [1, Theorem 1.4.10].

(1.17) Let $(D_{U,\mathrm{nat}}^{\leq 0}, D_{U,\mathrm{nat}}^{\geq 0})$ {resp. $(D_{\mathrm{nat}}^{\leq 0}, D_{\mathrm{nat}}^{\geq 0})$} be the natural *t*-structure of $D^b(\mathrm{mod}(\Lambda_U, \Omega_U))$ {resp. $D^b(\mathrm{mod}(\Lambda, \Omega))$}. For the definition of natural *t*-structure see [1, 1.3.2]. Let $(D_{F,\mathrm{nat}}^{\leq 0}, D_{F,\mathrm{nat}}^{\geq 0})$ be the natural *t*-structure of $D^b(\mathrm{mod}\, k)$.

Remark 1.17a. The functors $i^!$, i_*, j^*, j_* are exact with respect to the natural t-structures (i.e. $D^{\leqslant 0}_{\ldots}$ goes into $D^{\leqslant 0}_{\ldots}$ and $D^{\geqslant 0}_{\ldots}$ goes into $D^{\geqslant 0}_{\ldots}$). This is an easy consequence of the definitions.

(1.18) In the context of Proposition 1.16b, if $D_U^{\leqslant 0} \subseteq D^{\leqslant 0}_{U,\mathrm{nat}}$ and $D_F^{\leqslant 0} = D^{\leqslant 0}_{F,\mathrm{nat}}$, then $\tilde{D}^{\leqslant 0}$ can also be described by

$$\tilde{D}^{\leqslant 0} = \{K \in D^{\leqslant 0}_{\mathrm{nat}} \mid j^*K \in D_U^{\leqslant 0}\}.$$

In addition one also has

$$\tilde{D}^{\leqslant 0} = \{K \in D^b(\mathrm{mod}\,(\Lambda, \Omega)) \mid j^*K \in D_U^{\leqslant 0} \text{ and } i^!K \in D_F^{\leqslant 0}\}.$$

These assertions will be proved below. But first we deduce from the last assertion and 1.16b (definition of $\tilde{D}^{\geqslant 0}$) the following

(1.18a) When $D_U^{\leqslant 0} \subseteq D^{\leqslant 0}_{U,\mathrm{nat}}$ and $D_F^{\leqslant 0} = D^{\leqslant 0}_{F,\mathrm{nat}}$, the functor $i^!$ is exact with respect to the t-structures $(\tilde{D}^{\leqslant 0}, \tilde{D}^{\geqslant 0})$ in $D^b(\mathrm{mod}\,(\Lambda, \Omega))$ and $(D_F^{\leqslant 0}, D_F^{\geqslant 0})$ in $D^b(\mathrm{mod}\,(k))$.

Proof of the assertions 1.18. Suppose $K \in D^b(\mathrm{mod}\,(\Lambda, \Omega))$ and $j^*K \in D_U^{\leqslant 0}$. We will show that $i^*K \in D_F^{\leqslant 0} \Leftrightarrow i^!K \in D_F^{\leqslant 0}$ and further when this is so, $K \in D^{\leqslant 0}_{\mathrm{nat}}$. The last assertion follows since $D^{\leqslant 0}_{\mathrm{nat}} = \{K \in D^b(\mathrm{mod}\,(\Lambda, \Omega)) \mid j^*K \in D^{\leqslant 0}_{U,\mathrm{nat}} \text{ and } i^!K \in D^{\leqslant 0}_{F,\mathrm{nat}}\}$. Recall the definitions of i^*K, $i^!K$ and R^+_α {see resp. 1.14, 1.12 and 1.3b}. Write $K = \{V_\mu, \ldots, V(\overrightarrow{\mu\nu}), \ldots\}$ in the notation of 1.3. Then we have a distinguished triangle

$$L \to i^!K \to i^*K \to L[1] \tag{1.19}$$

in $D^b(\mathrm{mod}\,k)$ where L is the left side of 1.3a (i.e. the direct sum of the parts of K over vertices adjacent to α). From the assumptions $j^*K \in D_U^{\leqslant 0}$ and $D_U^{\leqslant 0} \subseteq D^{\leqslant 0}_{U,\mathrm{nat}}$, it follows that $L \in D^{\leqslant 0}_{F,\mathrm{nat}}$ $(= D_F^{\leqslant 0}$ by assumption) and aportioti $L[1] \in D_F^{\leqslant 0}$. The distinguished triangles $(L, i^!K, i^*K)$ and $(i^!K, i^*K, L[1])$ (1.19) imply that $i^!K \in D_F^{\leqslant 0} \Leftrightarrow i^*K \in D_F^{\leqslant 0}$, this completes the proof of the second assertion in 1.18. In particular, one deduces that $\tilde{D}^{\leqslant 0} \subseteq D^{\leqslant 0}_{\mathrm{nat}}$. Conversely suppose $K \in D^{\leqslant 0}_{\mathrm{nat}}$ and $j^*K \in D_U^{\leqslant 0}$. Then $i^!K \in D^{\leqslant 0}_{F,\mathrm{nat}} (= D_F^{\leqslant 0})$ and hence also $i^*K \in D_F^{\leqslant 0}$. Then, by what is already proved $K \in \tilde{D}^{\leqslant 0}$. This completes the proof of the first assertion in 1.18.

2. The t-structure associated to a data

Applying the previous constructions inductively, we will produce some t-structures in $D^b(\mathrm{mod}\,(\Lambda, \Omega))$. Recall that we assume that (Λ, Ω) does not have oriented cycles. In addition we assume that there is at most one arrow between any two vertices. Let n be the cardinality of Λ.

DEFINITION 2.1

A *data* in (Λ, Ω) is a collection of n sequences $\alpha_{p1}, \alpha_{p2}, \ldots, \alpha_{pv_p}$ $(1 \leqslant p \leqslant n, 1 \leqslant v_p)$ of not necessarily distinct vertices of (Λ, Ω). We assume that they satisfy the following conditions: α_{11} is a sink of (Λ, Ω); α_{12} is a sink of $(\Lambda, s_{11}\Omega)$ {here and in the sequel, $s_{p\mu}\Omega$ is abbreviation for $s_{\alpha_{p\mu}}\Omega$}$\ldots$; $\alpha_{1\mu}$ is a sink of $(\Lambda, s_{1(\mu-1)}\cdots s_{12}s_{11}\Omega)$ $(1 \leqslant \mu \leqslant v_1)$. Define

quivers (Λ_1, Ω_1) and (Λ_1, Ω_1') by $(\Lambda_1, \Omega_1) = (\Lambda, \Omega)$ and $(\Lambda_1, \Omega_1') = (\Lambda, s_{1(v_1-1)} \cdots s_{11}\Omega)$. Let (Λ_p, Ω_p) and (Λ_p, Ω_p') $(1 \leqslant p \leqslant n)$ be the quivers defined inductively as follows. If (Λ_q, Ω_q) and (Λ_q, Ω_q') have been defined $(\Lambda_{q+1}, \Omega_{q+1})$ is obtained by deleting α_{q,v_q} and all arrows to it from (Λ_q, Ω_q'). We assume that for each p $(1 \leqslant p \leqslant n)$, α_{p1} is a sink of (Λ_p, Ω_p), α_{p2} is a sink of $(\Lambda_p, s_{p1}\Omega_p)$ ... etc. .... α_{pv_p} is a sink of $(\Lambda_p, s_{p(v_p-1)} \cdots s_{p2}s_{p1}\Omega_p)$. We then set

$$(\Lambda_{q+1}, \Omega_{q+1}') = (\Lambda_{q+1}, s_{q+1(v_{q+1}-1)} \cdots s_{q+1,2}s_{q+1,1}\Omega_{q+1}).$$

(2.2) The natural t-structure in $D^b(\mathrm{mod}(\Lambda_q, \Omega_q'))$ {resp. $D^b(\mathrm{mod}(\Lambda_{q+1}, \Omega_{q+1}))$} will be denoted by $(D_q'^{\leqslant 0}, D_q'^{\geqslant 0})$ {resp. $D_{q+1}^{\leqslant 0}, D_{q+1}^{\geqslant 0}$)}. We will now apply the constructions of the previous section taking $(\Lambda, \Omega) = (\Lambda_q, \Omega_q')$ and $(\Lambda_U, \Omega_U) = (\Lambda_{q+1}, \Omega_{q+1})$. More particularly, remark 1.17a implies the following proposition.

{The functors $i^*, i_*, i^!, j_!, j^*, j_*, \tilde{i}_*, \tilde{i}^!, \tilde{j}^*, \tilde{j}_*$ defined in § 1 are denoted by the same symbols in the situation $(\Lambda, \Omega) = (\Lambda_q, \Omega_q')$ and $(\Lambda_U, \Omega_U) = (\Lambda_{q+1}, \Omega_{q+1})$}.

PROPOSITION 2.3

Let $(\tilde{D}_{q+1}^{\leqslant 0}, \tilde{D}_{q+1}^{\geqslant 0})$ *be any t-structure in* $D^b(\mathrm{mod}(\Lambda_{q+1}, \Omega_{q+1}))$ *such that* $\tilde{D}_{q+1}^{\leqslant 0} \subseteq D_{q+1}^{\leqslant 0}$. *Define* $\tilde{D}_q^{\leqslant 0} = \{X \in D_q'^{\leqslant 0} \mid j^*X \in \tilde{D}_{q+1}^{\leqslant 0}\}$ *and* $\tilde{D}_q^{\geqslant 0} = \tilde{D}_q^{\geqslant 1}[1]$ *where* $\tilde{D}_q^{\geqslant 1} = \{X \in D^b(\mathrm{mod}(\Lambda_q, \Omega_q')) \mid \mathrm{Hom}(Y, X) = 0,\ \forall Y \in \tilde{D}_q^{\leqslant 0}\}$. *Then* $(\tilde{D}_q^{\leqslant 0}, \tilde{D}_q^{\geqslant 0})$ *is a t-structure in* $D^b(\mathrm{mod}(\Lambda_q, \Omega_q'))$, *such that* $\tilde{D}_q^{\leqslant 0} \subseteq D_q'^{\leqslant 0}$.

{Recall from [1, 1.3.4] that if $(D_{\cdot\cdot}^{\leqslant 0}, D_{\cdot\cdot}^{\geqslant 0})$ is a t-structure then $D_{\cdot\cdot}^{\geqslant 1}$ is the right orthogonal of $D_{\cdot\cdot}^{\leqslant 0}$}.

Remark 2.4. In the context of Proposition 2.3, the functor $j_*: D^b(\mathrm{mod}(\Lambda_{q+1}, \Omega_{q+1})) \to D^b(\mathrm{mod}(\Lambda_q, \Omega_q'))$ is exact, with respect to the t-structures $(\tilde{D}_{q+1}^{\leqslant 0}, \tilde{D}_{q+1}^{\geqslant 0})$ in $D^b(\mathrm{mod}(\Lambda_{q+1}, \Omega_{q+1}))$ and $(\tilde{D}_q^{\leqslant 0}, \tilde{D}_q^{\geqslant 0})$ in $D^b(\mathrm{mod}(\Lambda_q, \Omega_q'))$. Indeed, $X \in \tilde{D}_{q+1}^{\leqslant 0} \Rightarrow X \in D_{q+1}^{\leqslant 0}$ (by assumption) $\Rightarrow j_*X \in D_q'^{\leqslant 0}$ (1.17a). But as $j^*j_*X \approx X$ (1.15b), Proposition 2.3 $\Rightarrow j_*X \in \tilde{D}_q^{\leqslant 0}$. This shows j_* is '*right*' exact (:= $j_*(D_{\cdot\cdot}^{\leqslant 0}) \subseteq D_{\cdot\cdot}^{\leqslant 0}$). But in the general context of 'recollement' j_* is always '*left*' exact (:= $D_{\cdot\cdot}^{\geqslant 0}$ goes into $D_{\cdot\cdot}^{\geqslant 0}$)—an easy fact [1, Prop. 1.4.16(i)]. Moreover, as already remarked (1.18a) in the context of Proposition 2.3, $i^!$ is exact. (Again, $i^!$ is only left exact in the general context of 'recollement' [1, Prop. 1.4.16(i)].)

(2.5) To apply Proposition 2.3 for the (descending) inductive construction of $(\tilde{D}_q^{\leqslant 0}, \tilde{D}_q^{\geqslant 0})$ for $1 \leqslant q \leqslant n$, we will transport $(\tilde{D}_q^{\leqslant 0}, \tilde{D}_q^{\geqslant 0})$ to $D^b(\mathrm{mod}(\Lambda_q, \Omega_q))$ using an isomorphism (described below) of $D^b(\mathrm{mod}(\Lambda_q, \Omega_q'))$ with $D^b(\mathrm{mod}(\Lambda_q, \Omega_q))$. Recalling 2.1 and using 1.6, we now have

Lemma 2.5a. For $1 \leqslant \mu < v_q$, $\alpha_{q\mu}$ *is a source of* $(\Lambda_q, s_{q\mu}s_{q\mu-1} \cdots s_{q1}\Omega_q)$. *Thus, the functor* $R_{q\mu}^- = R_{\alpha_{q\mu}}^- : D^b(\mathrm{mod}(\Lambda_q, s_{q\mu} \cdots s_{q1}\Omega_q)) \to D^b(\mathrm{mod}(\Lambda_q, s_{q(\mu-1)} \cdots s_{q1}\Omega_q))$ *is an equivalence of triangulated categories. Let* $R_q' = R_{q1}^- \circ R_{q2}^- \circ \cdots \circ R_{q(v_q-1)}^-$. *Then the functor* R_q' *is an equivalence of* $D^b(\mathrm{mod}(\Lambda_q, \Omega_q'))$ *with* $D^b(\mathrm{mod}(\Lambda_q, \Omega_q))$.

We will recall briefly the notation in 1.5 where we defined R_α^- to show

(2.6) R_α^- is right exact with respect to the natural t-structures.

Proof. If V''_α is the cone on the morphism 1.5a, we have a distinguished triangle

$$V'_\alpha \to \bigoplus_{\overrightarrow{\alpha\beta} \in E} V'_\beta \to V''_\alpha \to V'_\alpha[1]$$

(with notation as in 1.5a). If $V' \in D'^{\leqslant 0}$ for the natural t-structure then the above triangle implies that $V''_\alpha \in D'^{\leqslant 0}$ for the natural t-structure of $D^b(\text{mod } k)$. If we recall how R^-_α is defined (1.5), it now ensues that $R^-_\alpha(V') \in D'^{\leqslant 0}$ for the natural t-structure of $D^b(\text{mod}(\Lambda, s_\alpha\Omega))$.

We can now conclude that the functor R'_q in Lemma 2.5a is right exact with respect to the natural t-structures, i.e.,

$$R'_q(D'^{\leqslant 0}_q) \subseteq D'^{\leqslant 0}_q. \tag{2.7}$$

In particular, if $\tilde{D}'^{\leqslant 0}_q$ is given by Proposition 2.3, then

$$R'_q(\tilde{D}'^{\leqslant 0}_q) \subseteq R'_q(D'^{\leqslant 0}_q) \subseteq D'^{\leqslant 0}_q. \tag{2.7a}$$

By abuse of notation we write $\tilde{D}'^{\leqslant 0}_q$ instead of $R'_q(\tilde{D}'^{\leqslant 0}_q)$ and regard $(\tilde{D}'^{\leqslant 0}_q, \tilde{D}'^{\geqslant 0}_q)$ as a t-structure in $D^b(\text{mod}(\Lambda_q, \Omega_q))$. Thus

$$\tilde{D}'^{\leqslant 0}_q \subseteq D'^{\leqslant 0}_q. \tag{2.7b}$$

Now we can apply Proposition 2.3 to produce a t-structure $(\tilde{D}'^{\leqslant 0}_{q-1}, \tilde{D}'^{\geqslant 0}_{q-1})$ in $D^b(\text{mod}(\Lambda_{q-1}, \Omega'_{q-1}))$ and so on...

(2.8) We can now define for each $q (1 \leqslant q \leqslant n)$ a t-structure $(\tilde{D}'^{\leqslant 0}_q, \tilde{D}'^{\geqslant 0}_q)$ in $D^b(\text{mod}(\Lambda_q, \Omega_q))$ as follows. For $(\tilde{D}'^{\leqslant 0}_n, \tilde{D}'^{\geqslant 0}_n)$ we take the image of the natural t-structure of $D^b(\text{mod}(\Lambda_n, \Omega'_n))$ under the equivalence R'_n (2.5a): $D^b(\text{mod}(\Lambda_n, \Omega'_n)) \to D^b(\text{mod}(\Lambda_n, \Omega_n))$. Then we inductively construct $(\tilde{D}'^{\leqslant 0}_{n-1}, \tilde{D}'^{\geqslant 0}_{n-1})$, $(\tilde{D}'^{\leqslant 0}_{n-2}, \tilde{D}'^{\geqslant 0}_{n-2}), \ldots,$ $(\tilde{D}'^{\leqslant 0}_1, \tilde{D}'^{\geqslant 0}_1)$ respectively in $D^n(\text{mod}(\Lambda_{n-1}, \Omega_{n-1}))$, $D^b(\text{mod}(\Lambda_{n-2}, \Omega_{n-2})), \ldots,$ $D^b(\text{mod}(\Lambda, \Omega))$.

(2.9) Finally, the t-structure $(\tilde{D}'^{\leqslant 0}_1, \tilde{D}'^{\geqslant 0}_1)$ is what we refer to in the sequel as the 't-structure associated to the data $\{\alpha_{p1}, \ldots, \alpha_{pv_p} | 1 \leqslant p \leqslant n\}$'.

(2.10a) Example. The natural t-structure of $D^b(\text{mod}(\Lambda, \Omega))$ can be obtained as the t-structure associated to a data in the following way (roughly, spreading out an admissible enumeration (1.2) of (Λ, Ω)).

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be an admissible enumeration of (Λ, Ω). Define a data $\{\alpha_{p1}, \ldots, \alpha_{pv_p} | 1 \leqslant p \leqslant n\}$ by $v_p = 1 \forall p$ and $\alpha_{p1} = \alpha_p$. In this example $(\Lambda_q, \Omega'_q) = (\Lambda_q, \Omega_q)$ for all q and R'_q is the identity functor. The t-structure associated to this data (2.9) is simply the natural t-structure of $D^b(\text{mod}(\Lambda, \Omega))$.

(2.10b) Example. Let (Λ, Ω) be the quiver $\overset{1}{\cdot} \to \overset{2}{\cdot} \to \overset{3}{\cdot}$ (this is a Dynkin quiver of type A_3). If α, β, γ are the simple roots corresponding to the vertices 1, 2, 3 respectively then the positive roots are $\alpha, \beta, \gamma, \alpha+\beta, \beta+\gamma$ and $\alpha+\beta+\gamma$. The indecomposable objects (up to isomorphism) of $\text{mod}(\Lambda, \Omega)$ are in 1-1 correspondence with the positive roots (according to Gabriel's observation). We denote these indecomposable objects by V_α, $V_\beta, V_\gamma, V_{\alpha+\beta}, V_{\beta+\gamma}, V_{\alpha+\beta+\gamma}$ respectively. Then the indecomposable objects of $D^b(\text{mod}(\Lambda, \Omega))$ (up to isomorphism) are (i) $\{V_\theta[i] | i \in \mathbb{Z}$ and $\theta \in (\alpha, \beta, \gamma, \alpha+\beta, \beta+\gamma, \alpha+\beta+\gamma)\}$ (see for

instance [5, 4.1]). If one restricts to $i \geqslant 0$ (resp. $i \leqslant 0$) in (i) then one gets the indecomposables of $D^{\leqslant 0}$ (resp. $D^{\geqslant 0}$) for the natural t-structure of $D^b(\mathrm{mod}(\Lambda, \Omega))$. The t-structure $(\tilde{D}_1^{\leqslant 0}, \tilde{D}_1^{\geqslant 0})$ in $D^b(\mathrm{mod}(\Lambda, \Omega))$ associated to the data $\{(3), (2, 1), (2)\}$ can be described as follows. The indecomposables of $\tilde{D}_1^{\leqslant 0}$ are of the form $V_\theta[i]$ where (i) either $\theta = \alpha, \gamma, \alpha+\beta, \alpha+\beta+\gamma$ and $i \geqslant 0$ or (ii) $\theta = \beta, \beta+\gamma$ and $i > 0$. The indecomposables of $\tilde{D}_1^{\geqslant 0}$ are of the form $V_\theta[i]$ where (i) either $\theta = \alpha, \gamma, \alpha+\beta, \beta+\gamma, \alpha+\beta+\gamma$ and $i \leqslant 0$, or (ii) $\theta = \beta$ and $i \leqslant 1$.

(2.11) We will now define some t-structures intermediate to $(\tilde{D}_1^{\leqslant 0}, \tilde{D}_1^{\geqslant 0})$ and the natural t-structure of $D^b(\mathrm{mod}(\Lambda, \Omega))$. They play a role in the proof of Theorem 7.1.

Fix a q, $1 \leqslant q \leqslant n$ and v such that $1 \leqslant v < v_q$. Define

(2.11a) $R'_{q,(v)}: D^b(\mathrm{mod}(\Lambda_q, s_{qv} \cdots s_{q1}\Omega_q)) \to D^b(\mathrm{mod}(\Lambda_q, \Omega_q))$ by $R'_{q,(v)} = R_{q1}^- \circ R_{q2}^- \circ \cdots \circ R_{qv}^-$ (2.5a). Thus for example,

(2.11b) $R'_{q,v_q-1} = R'_q$ (2.5a) in our earlier notation. Define $(\tilde{D}^{\leqslant 0}_{q,(v)}, \tilde{D}^{\geqslant 0}_{q,(v)})$ in $D^b(\mathrm{mod}(\Lambda_q, \Omega_q))$ to be the image of the natural t-structure of $D^b(\mathrm{mod}(\Lambda_q, s_{qv} \cdots s_{q1}\Omega_q))$ in $D^b(\mathrm{mod}(\Lambda_q, \Omega_q))$ under the isomorphism $R'_{q,(v)}$ and $\tilde{C}_{q,(v)}$ its heart. Starting with this $(\tilde{D}^{\leqslant 0}_{q,(v)}, \tilde{D}^{\geqslant 0}_{q,(v)})$ apply the inductive construction of Proposition 2.3 to obtain new t-structures $(\tilde{D}^{\leqslant 0}_{q-1}, \tilde{D}^{\geqslant 0}_{q-1}), \ldots, (\tilde{D}_1^{\leqslant 0}, \tilde{D}_1^{\geqslant 0})$ in $D^b(\mathrm{mod}(\Lambda_{q-1}, \Omega_{q-1})), \ldots, D^b(\mathrm{mod}(\Lambda_1, \Omega_1))$ respectively.

(2.11c) We will use the notation $(\hat{D}^{\leqslant 0}_{q,(v)}, \hat{D}^{\geqslant 0}_{q,(v)})$ to denote the t-structure $(\tilde{D}_1^{\leqslant 0}, \tilde{D}_1^{\geqslant 0})$ obtained in this way. In this notation, the t-structure associated to the data $\{\alpha_{p1}, \ldots, \alpha_{pv_p} \mid 1 \leqslant p \leqslant n\}$ is $(\hat{D}^{\leqslant 0}_{n,(v_n-1)}, \hat{D}^{\geqslant 0}_{n,(v_n-1)})$. One has the obvious inclusion relations

$$\hat{D}^{\leqslant 0}_{q,(v)} \subseteq \hat{D}^{\leqslant 0}_{q,(v-1)}, \hat{D}^{\leqslant 0}_{q,(v)} \subseteq \hat{D}^{\leqslant 0}_{q-1,(\mu)} \tag{2.11d}$$

and also $\hat{D}^{\geqslant 0}_{q,(v)} \supseteq \hat{D}^{\geqslant 0}_{q,(v-1)}$, $\hat{D}^{\geqslant 0}_{q,(v)} \supseteq \hat{D}^{\geqslant 0}_{q-1,\mu}$.

Remark (2.12). Let P be the simple projective of $\mathrm{mod}(\Lambda_q, s_{q,v-1} \cdots s_{q1}\Omega_q)$ corresponding to the sink $\alpha_{q,v}$. Then, the indecomposables of $\tilde{D}^{\leqslant 0}_{q,(v)}$ are obtained by dropping the isomorphism class of $R'_{q,(v-1)}(P)$ from the indecomposables of $\tilde{D}^{\leqslant 0}_{q,(v-1)}$.

3. A necessary and sufficient condition

If $(D^{\leqslant 0}, D^{\geqslant 0})$ is any t-structure in $D^b(\mathrm{mod}(\Lambda, \Omega))$ denote by $\mathscr{G}$ the full subcategory $D^{\leqslant 0} \cap D^{\geqslant 0}$. It is an abelian category called the 'heart' of the t-structure [1, 1.3.1]. If $D^b(\mathscr{G})$ denotes the bounded derived category of $\mathscr{G}$, then a functor 'real' ('realization'): $D^b(\mathscr{G}) \to D^b(\mathrm{mod}(\Lambda, \Omega))$ is defined in [1, 3.1.10]. In this article, we are concerned with the t-structures for which the functor real is an equivalence of triangulated categories.

(3.1) Recall the functors $j_!, j_*: D^b(\mathrm{mod}(\Lambda_{q+1}, \Omega_{q+1})) \to D^b(\mathrm{mod}(\Lambda_q, \Omega'_q))$ and $j_*: D^b(\mathrm{mod}(\Lambda_{q+1}, \Omega_{q+1})) \to D^b(\mathrm{mod}(\Lambda_q, s_{qv_q}\Omega'_q))$ (2.2 and 1.8). We now define functors

$$\rho_q: D^b(\mathrm{mod}(\Lambda_{q+1}, \Omega'_{q+1})) \to D^b(\mathrm{mod}(\Lambda_q, \Omega'_q)) \text{ by}$$
$$\rho_q = j_! R'_{q+1}, \text{ (2.5a).} \tag{3.1a}$$

For each $p (1 \leqslant p \leqslant n)$ define $\mathscr{G}'(p) =$ the heart of the natural t-structure of

$D^b(\mathrm{mod}(\Lambda_p, \Omega'_p))$. For simplicity of notation $R'_p(\mathscr{G}'(p))$ will also be denoted by $\mathscr{G}'(p)$.

Let $\tilde{\mathscr{G}}_1$ denote the heart of $(\tilde{D}_1^{\leqslant 0}, \tilde{D}_1^{\geqslant 0})$ constructed in §2 (2.9).

A necessary and sufficient condition (effaceability) for the functor real to be an isomorphism is given in [1, Prop. 3.1.16] (see 3.4 below). For the t-structure $(\tilde{D}_1^{\leqslant 0}, \tilde{D}_1^{\geqslant 0})$ (2.9) we will derive a consequence of this condition. In this form, though not very transparent, it is easy to verify in practice as will become clear with a few examples (5.1 and 5.2).

(3.2) Write $V(q)$ for the simple object of $\mathrm{mod}(\Lambda_q, \Omega'_q)$ corresponding to the vertex α_{q,v_q} (i.e. we have a one-dimensional vector space over the vertex α_{q,v_q} and null vector space over other vertices).

Theorem 3.3. *The functor real: $D^b(\tilde{\mathscr{G}}_1) \to D^b(\mathrm{mod}(\Lambda, \Omega))$ is an equivalence of triangulated categories if and only if*

For each p, q with $1 \leqslant p \leqslant q < n$ either $\rho_p \circ \rho_{p+1} \circ \cdots \circ \rho_q(V(q+1))$ belongs to $\mathscr{G}'(p)$ or equals $j_* \circ R'_{p+1} \circ (\rho_{p+1} \circ \rho_{p+2} \circ \cdots \circ \rho_q)(V(q+1))$. {Here j_* is functor (1.8a, 2.2) $D^b(\mathrm{mod}(\Lambda_{p+1}, \Omega_{p+1})) \to D^b(\mathrm{mod}(\Lambda_p, \Omega'_p))$}. (3.3a)

When this condition is satisfied up to isomorphism the objects $R'_1 \circ (\rho_1 \circ \cdots \circ \rho_{q-1})(V(q))$, $(1 \leqslant q \leqslant n)$ are precisely all the indecomposable projectives of $\tilde{\mathscr{G}}_1$.

Proof. By (1, Prop. 3.1.16).

(3.4) A necessary and sufficient condition for real: $D^b(\tilde{\mathscr{G}}_1) \to D^b(\mathrm{mod}(\Lambda, \Omega))$ to be an equivalence is the following:

Given $A, B \in \tilde{\mathscr{G}}_1$ and $f \in \mathrm{Hom}(A, B[n])$ where $n > 0$, $\exists$ a monomorphism (in $\tilde{\mathscr{G}}_1$) $B \to C$ such that f has image zero under the canonical map $\mathrm{Hom}(A, B[n]) \to \mathrm{Hom}(A, C[n])$.

Remark (3.5) Observe that condition 3.4 is equivalent to "the property 3.4 holds even if one only assumes $A \in \tilde{D}_1^{\geqslant 0}$."

Indeed, first if $A = A'[-m]$ where $m \geqslant 0$ then $\mathrm{Hom}(A, B[n]) \approx \mathrm{Hom}(A', B[m+n])$ and so the property 3.4 for A' implies the property for A. Next, suppose $A \in \tilde{D}_1^{\geqslant 0} \cap \tilde{D}_1^{\leqslant m+1}$. Then, one has a truncation triangle $\tau_{\leqslant m}(A) \to A \to \tau_{\geqslant m+1}(A)$ where $\tau_{\leqslant m}(A) \in D^{\geqslant 0} \cap D^{\leqslant m}$ and $\tau_{\geqslant m+1}(A) \in \tilde{\mathscr{G}}_1[-m-1]$. A suitable induction argument (see the proof of 3.18) completes the proof.

(3.6) Fix p, $(1 \leqslant p \leqslant n)$. The n-p sequences $\alpha_{q1}, \ldots, \alpha_{qn_q} (p+1 \leqslant q \leqslant n)$ give rise to a t-structure in $D^b(\mathrm{mod}(\Lambda_{p+1}, \Omega_{p+1}))$ via the construction of §2. Clearly, this coincides with what was denoted $(\tilde{D}_{p+1}^{\leqslant 0}, \tilde{D}_{p+1}^{\geqslant 0})$ in the inductive construction of §2 (2.8). The composite of the functors $D^b(\mathrm{mod}(\Lambda_{p+1}, \Omega_{p+1})) \xrightarrow{j_*} D^b(\mathrm{mod}(\Lambda_p, \Omega'_p)) \xrightarrow{R'_p} D^b(\mathrm{mod}\ (\Lambda_p, \Omega_p)) \xrightarrow{j_*} D^b(\mathrm{mod}\ (\Lambda_{p-1}, \Omega'_{p-1})) \xrightarrow{R'_{p-1}} D^b(\mathrm{mod}\ (\Lambda_{p-1}, \Omega_{p-1})) \xrightarrow{j_*} \cdots \to D^b(\mathrm{mod}(\Lambda_1, \Omega_1))$ is seen to be exact (using Remark 2.4) with respect to the t-structures $(\tilde{D}_{p+1}^{\leqslant 0}, \tilde{D}_{p+1}^{\geqslant 0})$ in $D^b(\mathrm{mod}(\Lambda_{p+1}, \Omega_{p+1}))$ and $(\tilde{D}_1^{\leqslant 0}, \tilde{D}_1^{\geqslant 0})$ in $D^b(\mathrm{mod}(\Lambda, \Omega))$. {To avoid a possible confusion, we point out here that even though R'_p is only right exact with respect to the natural t-structures (by 2.7) it is exact with respect to $(\tilde{D}_p^{\leqslant 0}, \tilde{D}_p^{\geqslant 0})$ in $D^b(\Lambda_p, \Omega'_p)$ and its transport by R'_p denoted by the same notation $(\tilde{D}_p^{\leqslant 0}, \tilde{D}_p^{\geqslant 0})$ in $D^b(\mathrm{mod}(\Lambda_p, \Omega_p))$}.

(3.6a) We denote this composite by $\prod_{p,*}$. Thus, $\prod_{p,*}: D^b(\mathrm{mod}(\Lambda_{p+1}, \Omega_{p+1})) \to D^b(\mathrm{mod}(\Lambda, \Omega))$ is exact with respect to $(\tilde{D}^{\leqq 0}_{p+1}, \tilde{D}^{\geqq 0}_{p+1})$ and $(\tilde{D}^{\leqq 0}_1, \tilde{D}^{\geqq 0}_1)$.

(3.7) We denote by $\tilde{\mathscr{G}}_p$ the heart of the t-structure $(\tilde{D}^{\leqq 0}_p, \tilde{D}^{\geqq 0}_p)$ in $D^b(\mathrm{mod}(\Lambda_p, \Omega'_p))$ and we use the same notation $\tilde{\mathscr{G}}_p$ for $R'_p(\tilde{\mathscr{G}}_p)$. By $\mathscr{G}(p)$ (resp. $\mathscr{G}'(p)$) we denote the heart of the natural t-structure of $D^b(\mathrm{mod}(\Lambda_p, \Omega_p))$ (resp. $D^b(\mathrm{mod}(\Lambda_p, \Omega'_p))$). Again $R'_p(\mathscr{G}'(p))$ will also be denoted by $\mathscr{G}'(p)$.

(3.8) By 1.10 (ii), $\mathrm{Hom}(j_*X, j_*Y) \approx \mathrm{Hom}(j^*j_*X, Y) \approx \mathrm{Hom}(X, Y)$. Thus, for $X, Y \in \tilde{\mathscr{G}}_{p+1}$, $\prod_{p,*}(X)$ and $\prod_{p,*}(Y)$ belong to $\tilde{\mathscr{G}}_1$ and $\mathrm{Hom}(\prod_{p,*}(X), \prod_{p,*}(Y)) \approx \mathrm{Hom}(X, Y)$ and $\mathrm{Hom}(\prod_{p,*}(X), \prod_{p,*}(Y)[n]) \approx \mathrm{Hom}(X, Y[n])$ for $n \in \mathbb{Z}$.

(3.9) Similarly the composite of the functors $D^b(\mathrm{mod}(\Lambda_1, \Omega_1)) \approx D^b(\mathrm{mod}(\Lambda_1, \Omega'_1)) \xrightarrow{j^*} D^b(\mathrm{mod}(\Lambda_2, \Omega_2)) \approx D^b(\mathrm{mod}(\Lambda_2, \Omega'_2)) \xrightarrow{j^*} \cdots \xrightarrow{j^*} D^b(\mathrm{mod}(\Lambda_{p+1}, \Omega_{p+1}))$ is exact with respect to $(\tilde{D}^{\leqq 0}_1, \tilde{D}^{\geqq 0}_1)$ and $(\tilde{D}^{\leqq 0}_{p+1}, \tilde{D}^{\geqq 0}_{p+1})$. Denote this composite functor by $\prod^*_p$. As it is exact, in particular, if $\Psi: A \to B$ is a monomorphism in $\tilde{\mathscr{G}}_1$, then $\prod^*_p \Psi: \prod^*_p A \to \prod^*_p B$ is a monomorphism in $\tilde{\mathscr{G}}_{p+1}$. We also observe that $\prod^*_p \circ \prod_{p,*} \approx$ identity.

Lemma 3.10. If real: $D^b(\tilde{\mathscr{G}}_1) \to D^b(\mathrm{mod}(\Lambda, \Omega))$ *is an equivalence then real:* $D^b(\tilde{\mathscr{G}}_{p+1}) \to D^b(\mathrm{mod}(\Lambda_{p+1}, \Omega_{p+1}))$ *is also an equivalence.*

Proof. Let $X, Y \in \tilde{\mathscr{G}}_{p+1}$ and $f \in \mathrm{Hom}(X, Y[n]), n > 0$. Let $\varphi: \prod_p Y \to M$ be a monomorphism in $\tilde{\mathscr{G}}_1$ such that $\prod_p(f)$ goes to zero under the canonical map $\mathrm{Hom}(\prod_p X, \prod_p Y[n]) \to \mathrm{Hom}(\prod_p X, M[n])$. Using Remark 3.9, we then see that $\prod^*_p(\varphi): \prod^*_p \prod_{p,*} Y (\approx Y) \to \prod^*_p M$ is a monomorphism and f goes to zero under $\mathrm{Hom}(X, Y[n]) \to \mathrm{Hom}(X, \prod^*_p M[n])$. Hence by [1, Prop. 3.1.16] real: $D^b(\tilde{\mathscr{G}}_{p+1}) \to D^b(\mathrm{mod}(\Lambda_{p+1}, \Omega_{p+1}))$ is an equivalence. q.e.d.

Proof of the necessity part in Theorem. By suitable induction hypothesis, we can assume the validity of the assertion in the theorem for (Λ_2, Ω_2). Using induction hypothesis and Lemma 3.10, we conclude that for $2 \leqq p \leqq q < n$ either $\rho_p \circ \rho_{p+1} \circ \cdots \circ \rho_q(V(q+1))$ belongs to $\mathscr{G}'(p)$ or equals $j_* R'_{p+1}(\rho_{p+1} \circ \cdots \circ \rho_q V(q+1))$ and furthermore $R'_2(V(2))$ and $R'_2(\rho_2 \circ \cdots \circ \rho_{q-1})(V(q))$ $(2 < q \leqq n)$ are all the indecomposable projectives of $\tilde{\mathscr{G}}_2$ (upto isomorphism).

(3.11) Recall the functors $j_*, j_!: D^b(\mathrm{mod}(\Lambda_2, \Omega_2)) \to D^b(\mathrm{mod}(\Lambda_1, \Omega'_1))$ and $D^b(\mathrm{mod}(\Lambda_1, \Omega'_1)) \underset{i_*}{\overset{i^!}{\rightleftarrows}} D^b(\mathrm{mod}\, k)$ (see 1.8a, c, 1.11, 1.12 and 2.2). For any $K \in D^b(\mathrm{mod}(\Lambda_1, \Omega'_1))$, we have a distinguished triangle

$$i_* i^! K \to K \to j_* j^* K \xrightarrow{d} i_* i^! K[1].$$

Let P be an indecomposable projective of $\tilde{\mathscr{G}}_2$. Taking $K = j_! P$ and using the isomorphism $j^* j_! P \approx P$, we have a distinguished triangle

$$i_* i^! j_! P \to j_! P \to j_* P \xrightarrow{d} i_* i^! j_! P[1]. \tag{3.12}$$

Since $j_!$ is right exact [1, 1.4.16 i], i_* is exact (*loc cit*) and $i^!$ is exact (1.18a). We conclude $i_*i^!(j_!P) \in D_F^{\leqslant 0}$ (1.18).

(3.13) We claim that in fact $i_*i^!(j_!P) \in \mathscr{G}_F (= D_F^{\leqslant 0} \cap D_F^{\geqslant 0})$.

To see this note that $j_!P$ and j_*P are both indecomposable objects of $D^b(\mathrm{mod}(\Lambda, \Omega))$. This follows, for example, from the assumption that P is indecomposable and the fact $\mathrm{Hom}(j_!P, j_!P) = \mathrm{Hom}(P, j^*j_!P) = \mathrm{Hom}(P, P)$ and $\mathrm{Hom}(j_*P, j_*P) = \mathrm{Hom}(j^*j_*P, P) = \mathrm{Hom}(P, P)$.

For simplicity let us write N for $i_*i^!j_!P$. Thus $N \in D_F^{\leqslant 0}$ and $N = H^\circ(N) \oplus \tau_{\leqslant -1}(N)$. The map $d: j_*P \to N[1]$ is a sum of two maps $d': j_*P \to H^\circ(N)[1]$ and $d'': j_*P \to (\tau_{\leqslant -1}(N)[1])$. If we show $d'' = 0$, then the distinguished triangle $H^\circ(N) \oplus \tau_{\leqslant -1}(N) \to j_!P \to j_*P \xrightarrow{d'+d''} H^\circ(N)[1] \oplus (\tau_{\leqslant -1}(N))[1]$ would imply $j_!P = \tau_{\leqslant -1}(N) \oplus N'$ where N' occurs in a distinguished triangle $H^\circ(N) \to N' \to j_*P \xrightarrow{d'} H^\circ(N)[1]$ (Note: $N' \neq 0$. For, applying j^* to the last triangle $0 \to j^*N' \to P \to 0$ is a distinguished triangle). Since $j_!P$ is indecomposable, it would follow $\tau_{\leqslant -1}N = 0$ and $N \approx H^\circ(N) \in \mathscr{G}_F$ as desired.

(3.13a) It remains to show $d'' = 0$. Indeed we will show that $\mathrm{Hom}(j_*P, Y) = 0$ for any $Y \in i_*D_F^{\leqslant -2}$. If not $\exists n \geqslant 2$, such that $\mathrm{Hom}(j_*P, V[n]) \neq 0$, where V is the unique simple object of $i_*\mathscr{G}_F (\approx \mathrm{mod}\, k)$. Let f be a nonzero element of $\mathrm{Hom}(j_*P, V[n])$. Since by assumption real: $D^b(\tilde{\mathscr{G}}_1) \to D^b(\mathrm{mod}(\Lambda, \Omega))$ is an equivalence and since j_*P and $V \in \tilde{\mathscr{G}}_1$, using [1, 3.1.16] $\exists$ a monomorphism $V \hookrightarrow M$ in $\tilde{\mathscr{G}}_1$ such that f goes to zero under the map $\mathrm{Hom}(j_*P, V[n]) \to \mathrm{Hom}(j_*P, M[n])$. We have a distinguished triangle

$$i_*i^!M \to M \to j_*j^*M, \tag{3.14}$$

which is in fact a short exact sequence $0 \to i_*i^!M \to M \to j_*j^*M \to 0$ in $\tilde{\mathscr{G}}_1$ as $i^!$ and j_* (and of course i_* and j^* also) are exact. (Here, we use [1, 1.2.2.1].) The monomorphism $V \hookrightarrow M$ factorizes as $V \hookrightarrow i_*i^!M \to M$; this follows from the observation $V \in i_*\mathscr{G}_F$ and $\mathrm{Hom}(i_*, \ldots, j_*, \ldots) = 0$. Applying $\mathrm{Hom}(j_*P, \ldots)$ to the distinguished triangle 3.14 we have a long exact sequence

$$\begin{aligned}
&\mathrm{Hom}(j_*P, i_*i^!M) \to \mathrm{Hom}(j_*P, M) \to \mathrm{Hom}(j_*P, j_*j^*M) \\
&\quad \to \mathrm{Hom}(j_*P, (i_*i^!M)[1]) \to \mathrm{Hom}(j_*P, M[1]) \\
&\quad \to \mathrm{Hom}(j_*P, (j_*j^*M)[1]) \to \mathrm{Hom}(j_*P, (i_*i^!M)[2]) \\
&\quad \to \mathrm{Hom}(j_*P, M[2]) \to \mathrm{Hom}(j_*P, (j_*j^*M)[2]).
\end{aligned} \tag{3.14a}$$

But, $\mathrm{Hom}(j_*P, (j_*j^*M)[l]) \approx \mathrm{Hom}(P, j^*M[l]) = 0$ for $l \geqslant 1$ as P is projective in $\tilde{\mathscr{G}}_2$. The long exact sequence implies $\mathrm{Hom}(j_*P, (i_*i^!M)[l]) \to \mathrm{Hom}(j_*P, M[l])$ is an isomorphism for $l \geqslant 2$. Thus f goes to zero under the map $\mathrm{Hom}(j_*P, V[n]) \to \mathrm{Hom}(j_*P, (i_*i^!M)[n])$. But this is impossible since the inclusion $V \hookrightarrow i_*i^!M$ actually splits ($V \hookrightarrow i_*i^!M$ is an inclusion in $\mathscr{G}_F \approx \mathrm{mod}\, k$).

We have now shown $d'' = 0$ and 3.13 is proved.

For simplicity let us identify $D^b(\mathrm{mod}(\Lambda_1, \Omega_1'))$ with $D^b(\mathrm{mod}(\Lambda_1, \Omega_1))$ using the equivalence R_1'.

We have now a short exact sequence $0 \to N \to j_! P \to j_* P \to 0$ in $\mathscr{G}_1$, where $N = i_* i^! j_! P \in i_* \mathscr{G}_F$.

If $N = 0$ then $j_! P \approx j_* P$ which is as asserted in the theorem. If $N \neq 0$, N is isomorphic to a finite sum of copies of V, the simple object corresponding to the vertex α_{1k_1} of (Λ_1, Ω_1'). Since α_{1k_1} is a sink of (Λ_1, Ω_1') V is a projective object of $\mathrm{mod}\,(\Lambda_1, \Omega_1')$. Here we identify $\mathrm{mod}\,(\Lambda_1, \Omega_1')$ to $\mathscr{G}'(1)$, the heart of the natural t-structure of $D^b(\mathrm{mod}\,(\Lambda_1, \Omega_1'))$.

(3.15) It will be shown below that if $X \in D^b(\mathrm{mod}\,(\Lambda_1, \Omega_1'))$ is indecomposable and $\mathrm{Hom}\,(V, X) \neq 0$ then $X \in \mathscr{G}'(1)$.

Thus, when $N \neq 0$, $j_! P \in \mathscr{G}'(1)$ which is as asserted in the theorem.

Except for 3.15 this completes the proof of the necessity part of the theorem. That $j_! P$ is projective in $\tilde{\mathscr{G}}_1$ follows from below as we assumed real: $D^b(\tilde{\mathscr{G}}_1) \to D^b(\mathrm{mod}\,(\Lambda, \Omega))$ is an equivalence. It remains to show 3.15.

(3.15a) It is known that for $A, B \in \mathrm{mod}\,(\Lambda, \Omega)$, (where, as usual, we assume Ω has no oriented cycles and it is a quiver without relations) $\mathrm{Ext}^l(A, B) = 0$ for $l > 1$. This leads to the fact that if $X \in D^b(\mathrm{mod}\,(\Lambda, \Omega))$, $X \approx \oplus (H^l(X))[-l]$. This can also be deduced from [5, Lemma 4.1]. This immediately implies that if $P \in \mathrm{mod}\,(\Lambda, \Omega))$ is projective and $X \in D^b(\mathrm{mod}\,(\Lambda, \Omega))$ is indecomposable then $\mathrm{Hom}\,(P, X) \neq 0 \Rightarrow X \in \mathrm{mod}\,(\Lambda, \Omega)$. This implies 3.15.

Sufficiency part in the theorem. We begin the proof of the sufficiency part by first showing

(3.16) Assume that the condition 3.3a in Theorem 3.3 is satisfied. Let $\rho_p \cdot \rho_{p+1} \cdots \rho_q(V(q+1))$ be the object of $D^b(\mathrm{mod}\,(\Lambda_p, \Omega_p'))$ as in Theorem 3.3. Then $\rho_p \cdot \rho_{p+1} \cdots \rho_q(V(q+1))$ belongs to $\tilde{\mathscr{G}}_p$ (3.7). Furthermore $\mathrm{Hom}\,(\rho_p \cdot \rho_{p+1} \cdots \rho_q(V(q+1)), X) = 0$, $\forall X \in \tilde{D}_p^{\leq -1}$.

(3.16a) Later, after we show real: $D^b(\tilde{\mathscr{G}}_p) \to D^b(\mathrm{mod}\,(\Lambda_p, \Omega_p'))$ is an equivalence, the above property can be rephrased as saying that $\rho_p \cdot \rho_{p+1} \cdots \rho_q(V(q+1))$ is projective in $\tilde{\mathscr{G}}_p$.

(As usual, we identify $D^b(\mathrm{mod}\,(\Lambda_p, \Omega_p'))$ with $D^b(\mathrm{mod}\,(\Lambda_p, \Omega_p))$ using the isomorphism R_p'.)

The following assertion should be considered for the case $p = q + 1$ of 3.16.

(3.16b) $R_p'(V(p))$ belongs to $\tilde{\mathscr{G}}_p$. Furthermore, $\mathrm{Hom}\,(R_p'(V(p)), X) = 0$, $\forall X \in \tilde{D}_p^{\leq -1}$.

$V(p) \in \mathrm{mod}\,(\Lambda_p, \Omega_p') \subseteq D_p'^{\leq 0}$ (2.2). By Proposition 2.3, $\tilde{D}_p^{\leq 0} = \{X \in D_p'^{\leq 0} \mid j^* X \in \tilde{D}_{p+1}^{\leq 0}\}$. As $V(p) \in D_p'^{\leq 0}$ and $j^* V(p) = 0$, we conclude $V(p) \in \tilde{D}_p^{\leq 0}$. But also $V(p) \in D_p'^{\geq 0} \subseteq \tilde{D}_p^{\geq 0}$. Hence $V(p) \in \tilde{D}_p^{\leq 0} \cap \tilde{D}_p^{\geq 0} = \tilde{\mathscr{G}}_p$. Thus $R_p'(V(p)) \in R_p'(\tilde{\mathscr{G}}_p)$, which, by our convention is also written $\tilde{\mathscr{G}}_p$. As α_{p,v_p} is a sink of (Λ_p, Ω_p') and $V(p)$ is the simple object of $\mathrm{mod}\,(\Lambda_p, \Omega_p')$ corresponding to that vertex, $V(p)$ is a projective object of $\mathrm{mod}\,(\Lambda_p, \Omega_p')$. Thus, $\mathrm{Hom}\,(V(p), X) = 0$, for $X \in D_p'^{\leq -1}$. Since $\tilde{D}_p^{\leq -1} \subseteq D_p'^{\leq -1}$, 3.16b follows.

We will prove 3.16 by descending induction on p. Our induction hypothesis is

(3.17) The assertion 3.16 is true for $p + 1$.

Write P for $R_{p+1}' \cdot \rho_{p+1} \cdots \rho_q(V(q+1))$.

Recall the functors

$$D^b(\mathrm{mod}\,(\Lambda_{p+1}, \Omega_{p+1}) \underset{j_* j_!}{\overset{j^*}{\rightleftarrows}} D^b(\mathrm{mod}\,(\Lambda_p, \Omega_p') \underset{i_*}{\overset{i^*, i^!}{\rightleftarrows}} D^b(\mathrm{mod}\,k).$$

By assumption 3.3a is satisfied. So either $j_!P = j_*P$, or $j_!P \in \mathscr{G}'(p)$. In the first case $j_!P \in \tilde{\mathscr{G}}_p$ since $P \in \tilde{\mathscr{G}}_{p+1}$ and j_* is exact (2.4) with respect to $(\tilde{D}_{p+1}^{\leqslant 0}, \tilde{D}_{p+1}^{\geqslant 0})$ and $(\tilde{D}_p^{\leqslant 0}, \tilde{D}_p^{\geqslant 0})$. In the second case $j_!P \in \mathscr{G}'(p) \subseteq \tilde{D}_p^{\geqslant 0}$. But also $j_!P \in \tilde{D}_p^{\leqslant 0}$ as $j_!$ is right exact [1, 1.4.16(i)]. Thus $j_!P \in \tilde{\mathscr{G}}_p$.

For any $X \in \tilde{D}_p^{\leqslant -1}$, in the distinguished triangle $i_* i^! X \to X \to j_* j^* X$ all the objects are in $\tilde{D}_p^{\leqslant -1}$ as $i^!$ and j_* are exact. From the long exact sequence obtained by applying $\mathrm{Hom}(j_!P, \ldots)$ to the above distinguished triangle and from the fact $\mathrm{Hom}(j_!, \ldots, i_*, \ldots) = 0$, we conclude that $\mathrm{Hom}(j_!P, X) \approx \mathrm{Hom}(j_!P, j_* j^* X)$. But, $\mathrm{Hom}(j_!P, j_* j^* X) = \mathrm{Hom}(j^* j_! P, j^* X) = \mathrm{Hom}(P, j^* X) = 0$ using induction hypothesis as $j^* X \in \tilde{D}_{p+1}^{\leqslant -1}$. Thus, $\mathrm{Hom}(j_!P, X) = 0$. But $j_!P = \rho_p \cdot \rho_{p+1} \cdots \rho_q(V(q+1))$. So 3.16 is proved.

Lemma 3.18. Suppose $0 \to A_1 \to A_2 \to A_3 \to 0$ is a short exact sequence in $\tilde{\mathscr{G}}_1$, or, more generally, suppose $A_1 \to A_2 \to A_3$ is any distinguished triangle in $D^b(\mathrm{mod}(\Lambda, \Omega))$. Let m be any integer $\geqslant 2$. Let $i = 1$ or 3. Suppose that for any $X \in \tilde{\mathscr{G}}_1$ and any $f \in \mathrm{Hom}(A_i, X[m])$ $\exists$ a monomorphism $X \to M$ in $\tilde{\mathscr{G}}_1$ such that f goes to zero under the natural map $\mathrm{Hom}(A_i, X[m]) \to \mathrm{Hom}(A_i, M[m])$. Then the same property also holds for $i = 2$.

Proof. Consider the exact sequence

$$\mathrm{Hom}(A_3, X[m]) \xrightarrow{\varphi_X} \mathrm{Hom}(A_2, X[m]) \xrightarrow{\psi_X} \mathrm{Hom}(A_1, X[m]).$$

Let $f \in \mathrm{Hom}(A_2, X[m])$. Let $X \to M'$ be a monomorphism such that $\varphi_X f$ goes to zero under $\mathrm{Hom}(A_1, X[m]) \to \mathrm{Hom}(A_1, M'[m])$. Consider the commutative diagram (where the rows are exact)

$$\begin{array}{ccccc} \mathrm{Hom}(A_3, X[m]) & \xrightarrow{\psi_X} & \mathrm{Hom}(A_2, X[m]) & \xrightarrow{\varphi_X} & \mathrm{Hom}(A_1, X[m]) \\ \eta_3 \downarrow & & \eta_2 \downarrow & & \eta_1 \downarrow \\ \mathrm{Hom}(A_3, M'[m]) & \xrightarrow{\psi_{M'}} & \mathrm{Hom}(A_2, M'[m]) & \xrightarrow{\varphi_{M'}} & \mathrm{Hom}(A_1, M'[m]). \end{array}$$

Since $\varphi_M \cdot \eta_2(f) = 0$, $\eta_2(f)$ comes from $\tilde{f} \in \mathrm{Hom}(A_3, M'[m])$. Let $M' \to M''$ be a monomorphism such that $\tilde{f}$ goes to zero under the map $\mathrm{Hom}(A_3, M'[m]) \to \mathrm{Hom}(A_3, M''[m])$. Then the monomorphism $X \to M''$ does the job for f. q.e.d.

In view of the lemma, it suffices to prove the sufficiency part for each simple object A. (One can easily show that in the abelian category $\tilde{\mathscr{G}}_1$, all objects have finite length.)

Lemma 3.19. The simple objects (up to isomorphism) of $\tilde{\mathscr{G}}_1$, are $S_1, S_2, \ldots, S_n$ where $S_p = R_1' \cdot j_* \cdot R_2' \cdot j_* \cdots R_{p-1}' \cdot j_* \cdot R_p' V(p)$, where $V(p)$ is the simple object of $\mathrm{mod}(\Lambda_p, \Omega_p')$ corresponding to the vertex α_{p, v_p}.

Proof. Let A be any object of $\tilde{\mathscr{G}}_1$. Consider the triangle $i_* i^! A \to A \to j_* j^* A$. Here $i_*, j_*, \ldots$ etc. refer to functors between $D^b(\mathrm{mod}(\Lambda_2, \Omega_2))$, $D^b(\mathrm{mod}(\Lambda_1, \Omega_1'))$ $D^b(\mathrm{mod}\, k)$. As we have seen before this actually comes from a short exact sequence $0 \to i_* i^! A \to A \to j_* j^* A \to 0$ in $\tilde{\mathscr{G}}_1$. Suppose A is a simple object then either $A \approx j_* j^* A$ and $j^* A$ is a simple object of $\tilde{\mathscr{G}}_2$ (j_* is exact!) or $A \approx i_* i^! A$. In the latter case, $A \approx V(1)$ (or, rather, $A \approx R_1' V(1)$). By iterating we see the validity of the assertion in the lemma. q.e.d.

Now let $i_*, j_*, \ldots$ etc. refer to the functors between $D^b(\mathrm{mod}(\Lambda_p, \Omega_p))$ $D^b(\mathrm{mod}(\Lambda_{p-1}, \Omega'_{p-1}))$ and $D^b(\mathrm{mod}\, k)$. Then we have a short exact sequence (in $\tilde{\mathscr{G}}_{p-1} \subseteq D^b(\mathrm{mod}(\Lambda_{p-1}, \Omega'_{p-1}))$ $0 \to i_* i^! j_! R'_p(V(p)) \to j_! R'_p(V(p)) \to j_* R'_p V(p) \to 0$ (3.12). Hence, we have a short exact sequence (in $\tilde{\mathscr{G}}_{p-1} \subseteq D^b(\mathrm{mod}(\Lambda_{p-1}, \Omega_{p-1}))$.

$$0 \to R'_{p-1} i_* i^! j_! R'_p V(p) \to R'_{p-1} j_! R'_p(V(p)) \to R'_{p-1} j_* R'_p(V(p)) \to 0. \tag{3.20}$$

Now apply $j_*: D^b(\mathrm{mod}(\Lambda_{p-1}, \Omega_{p-1})) \to D^b(\mathrm{mod}(\Lambda_{p-2}, \Omega'_{p-2}))$ (which is exact with respect to the t-structures $(\tilde{D}^{\leqslant 0}_{p-1}, \tilde{D}^{\geqslant 0}_{p-1})$ and $(\tilde{D}^{\leqslant 0}_{p-2}, \tilde{D}^{\geqslant 0}_{p-2})$. Thus, we have a surjection

$$j_* R'_{p-1} j_! R'_p V(p) \twoheadrightarrow j_* R'_{p-1} j_* R'_p V(p). \tag{3.21}$$

But we also have a short exact sequence (3.12)

$$0 \to i_* i^! j_! K \to j_! K \to j_* K \to 0,$$

where $K = R'_{p-1} j_! R'_p V(p)$, which gives a surjection

$$j_! R'_{p-1} j_! R'_p V(p) \twoheadrightarrow j_* R'_{p-1} j_! R'_p V(p). \tag{3.22}$$

Composing 3.22 and 3.21 we get a surjection

$$j_! R'_{p-1} j_! R'_p(V(p)) \twoheadrightarrow j_* R'_{p-1} j_* R'_p V(p)$$

and also

$$R'_{p-2} j_! R'_{p-1} j_! R'_p(V(p)) \twoheadrightarrow R'_{p-2} j_* R'_{p-1} j_* R'_p V(p).$$

Iterating we get a surjection

$$R'_1 j_! R'_2 j_! \cdots R'_{p-1} j_! R'_p(V(p)) \to R'_1 j_* R'_2 j_* \cdots R'_{p-1} j_* R'_p(V(p)).$$

Recalling our notation (3.1a, 3.19) this is the same as a surjection

$$R'_1(\rho_1 \circ \rho_2 \cdots \circ \rho_{p-1})(V(p)) \to S_p. \tag{3.23}$$

If K is the kernel of 3.23, then we have an exact sequence

$$0 \to K \to R'_1(\rho_1 \cdot \rho_2 \cdots \rho_{p-1})(V(p)) \to S_p \to 0.$$

(3.23a) Write P_p for the middle term. Thus we have a short exact sequence

$$0 \to K \to P_p \to S_p \to 0.$$

Now we will show the property

(3.24) "If k is an integer $\geqslant 1$, $X \in \tilde{\mathscr{G}}_1$ and $f \in \mathrm{Hom}(S_p, X[k])$ $(1 \leqslant p \leqslant n)$ then there exists a monomorphism $X \hookrightarrow M$ in $\tilde{\mathscr{G}}_1$ such that f goes to zero under $\mathrm{Hom}(S_p, X[k]) \to \mathrm{Hom}(S_p, M[k])$" by induction on k. For $k = 1$, the condition is satisfied (without any hypothesis). The morphism f occurs in a distinguished triangle

$$X \to M \to S_p \xrightarrow{f} X[1].$$

Since $X, S_p \in \tilde{\mathscr{G}}_1$, we conclude that $M \in \tilde{\mathscr{G}}_1$ and $0 \to X \to M \to S_p \to 0$ is a short exact sequence. The inclusion $X \to M$ does the job (the composite of adjacent morphisms in a triangle is zero; $S_p \to X[1] \to M[1]$ is a triangle).

(3.25) Now assume 3.24 is true for $1 \leqslant k \leqslant l$. Then by Lemma 3.18, 3.24 is true with S_p replaced by any arbitrary $K \in \tilde{\mathscr{G}}_1$.

Applying $\mathrm{Hom}(.., X[l+1])$ to the triangle $K \to P_p \to S_p \to K[1] \to P_p[1]$, we get an exact sequence

$$\mathrm{Hom}(P_p[1], X[l+1]) \to \mathrm{Hom}(K[1], X[l+1])$$

$$\to \mathrm{Hom}(S_p, X[l+1]) \to \mathrm{Hom}(P_p, X[l+1]).$$

But $\mathrm{Hom}(P_p[1], X[l+1]) \approx \mathrm{Hom}(P_p, X[l]) = 0$ by 3.16 (see notation in 3.23a) and $\mathrm{Hom}(P_p, X(l+1]) = 0$ also for the same reason. Thus, we have an isomorphism $\mathrm{Hom}(K[1], X[l+1]) \to \mathrm{Hom}(S_p, X[l+1])$. Let $f \in \mathrm{Hom}(S_p, X[l+1])$. Choose $\tilde{f} \in \mathrm{Hom}(K[1], X[l+1])$ lying over f. As $\mathrm{Hom}(K[1], X[l+1]) \approx \mathrm{Hom}(K, X[l])$, by induction hypothesis 3.25 $\exists$ monomorphism $X \to M$ which annihilates $\tilde{f}$. But we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(K[1], X[l+1]) & \xrightarrow{\sim} & \mathrm{Hom}(S_p, X[l+1]) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(K[1], M[l+1]) & \xrightarrow{\sim} & \mathrm{Hom}(S_p, M[l+1]) \end{array}$$

and clearly

$$\begin{array}{ccc} \tilde{f} & \mapsto & f \\ \downarrow & & \downarrow \\ 0 & \mapsto & 0 \end{array}$$

This completes the proof of the sufficiency part of Theorem 3.3.

In particular, as remarked in 3.16a, the property 3.16 now implies that $R_1'(\rho_1 \cdot \rho_2 \cdots \rho_{p-1})(V(p))$ $(1 \leqslant p \leqslant n)$ are projective objects in $\tilde{\mathscr{G}}_1$. By Lemma 3.19 and 3.23, these indecomposable projective objects are enough to cover all the simple objects of $\tilde{\mathscr{G}}_1$. Hence they are all the indecomposable projectives.

This finishes the proof of Theorem 3.3. We illustrate with some examples in § 5.

4. A property of intermediate t-structures

The purpose of this section is to show that for a given data 2.1 if real: $D^b(\tilde{\mathscr{G}}_1) \to D^b(\mathrm{mod}(\Lambda, \Omega))$ is an equivalence, then it is an equivalence also for the intermediate t-structures defined in 2.11. The proof is mostly a repetition of what we saw in the necessity part of Theorem 3.3.

Fix a q and v as in 2.11. We freely use the notation introduced there.

(4.1) We remark that the t-structure $(\hat{D}^{\leqslant 0}_{q,(v)}, \hat{D}^{\geqslant 0}_{q,(v)})$ in $D^b(\mathrm{mod}(\Lambda, \Omega))$ (cf. 2.11c) can

ctually be obtained as the one associated (2.9) to a data $\{\beta_{p1},\ldots,\beta_{pv'_p}|\,1\leqslant p\leqslant n\}$ efined as follows: Choose any admissible enumeration (1.2) $\gamma_1,\ldots,\gamma_{n-q+1}$ of $_q, s_{qv}\cdots s_{q1}\Omega_q)$ with $\gamma_1=\alpha_{q,v+1}$. We set $v'_p=v_p$ for $1\leqslant p<q$ and $\beta_{p\mu}=\alpha_{p\mu}$ for $1\leqslant p<q$ nd $1\leqslant\mu\leqslant v_p$. Put $v'_q=v+1$, $\beta_{q\mu}=\alpha_{q\mu}$ for $1\leqslant\mu\leqslant v+1$. Put $v'_p=1$ for $q<p\leqslant n$ and $_1=\gamma_{p-q+1}$.

When we try to do the construction of §2 with the above β-data, the admissible umeration $\gamma_1,\ldots,\gamma_{n-q+1}$ ensures (because of 2.10a) that at level q of the inductive onstruction of §2, we get the natural t-structure of $D^b(\mathrm{mod}\,(\Lambda_q, s_{qv}\cdots s_{q1}\Omega_q))$. The maining steps iterate on this using Proposition 2.3. These are exactly the steps which oduced $(\hat{D}^{\leqslant 0}_{q,(v)}, \hat{D}^{\geqslant 0}_{q,(v)})$ in 2.11.

ROPOSITION 4.2

ppose real: $D^b(\tilde{\mathscr{G}}_1)\to D^b(\mathrm{mod}\,(\Lambda,\Omega))$ *is an equivalence. Then property* 3.3a *holds for the data in* 4.1.

roof. The case $q=1$ is trivial. So let $q>1$. Using suitable induction hypothesis we can sume the validity of the proposition for the data $\{\beta_{p1},\ldots,\beta_{pv'_p}|\,2\leqslant p\leqslant n\}$ in (Λ_2,Ω_2). et $\tilde{\mathscr{G}}_{\beta 2}$ denote the heart of the t-structure in $D^b(\mathrm{mod}\,(\Lambda_2,\Omega_2))$ associated to this data nd let P be an indecomposable projective in $\tilde{\mathscr{G}}_{\beta 2}$. We have a distinguished triangle .12)

$$i_*i^!(j_!P)\to j_!P\to j_*P\to i_*i^!(j_!P)[1],$$

here the functors are as in 3.11. (Incidentally, the quivers (Λ_2,Ω_2), (Λ_1,Ω'_1) the nctors $j_!$, j^*, j_*, i^*, i_*, $i^!$ etc., appearing in the constructions with the β-data coincide th the corresponding objects for the α-data. This is so since $q>1$.) For the same asons as explained following 3.12, $i_*i^!j_!(P)\in D_F^{\leqslant 0}$ and similar to 3.13 we wish to nclude

$$i_*i^!j_!(P)\in\mathscr{G}_F. \tag{4.3}$$

he proof of this part, as detailed below, is similar to that of 3.13, if we use 3.5. Again, we duce to showing $\mathrm{Hom}\,(j_*P, Y)=0$ for $Y\in i_*D_F^{\leqslant -2}$ (by the same arguments as eceding 3.13a). If $\mathrm{Hom}\,(j_*P, Y)\neq 0$ for some $Y\in i_*D_F^{\leqslant -2}$, $\exists n\geqslant 2$ such that om $(j_*P, V[n])\neq 0$ where V is the unique simple object of $i_*\mathscr{G}_F(\approx \mathrm{mod}\,k)$. Let f be a nzero element of $\mathrm{Hom}\,(j_*P, V[n])$. As $j_*P\in\hat{D}^{\geqslant 0}_{q,(v)}$ (using 2.4 for the β-data) and as $^{\geqslant 0}_{,(v)}\subseteq\tilde{D}_1^{\geqslant 0}$ (2.11d), it follows $j_*P\in\tilde{D}_1^{\geqslant 0}$. Using 3.5, $\exists$ a monomorphism $V\hookrightarrow M$ in $\tilde{\mathscr{G}}_1$ ch that f goes to zero under the map $\mathrm{Hom}\,(j_*P, V[n])\to\mathrm{Hom}\,(j_*P, M[n])$. We then ve the long exact sequence 3.14a and we still have $\mathrm{Hom}\,(j_*P,\ (j_*j^*M)[l])$ $\mathrm{Hom}\,(P, j^*M[l])=0$ as P is projective in $\tilde{\mathscr{G}}_{\beta 2}$ and $j^*M\in\tilde{D}_2^{\leqslant 0}\subseteq\tilde{D}_{\beta 2}^{\leqslant 0}$ (2.11d). After s, the same arguments which follow 3.14 show $i_*i^!j_!P\in\mathscr{G}_F$ and finally that $j_!P=j_*P$ $j_!P\in\mathscr{G}'(1)$. This completes the proof of 4.2.

OROLLARY 4.4

real: $D^b(\tilde{\mathscr{G}}_1)\to D^b(\mathrm{mod}\,(\Lambda,\Omega))$ *is an equivalence and* $\hat{\mathscr{G}}_{q,(v)}$ *denotes the heart of the t-ucture* $(\hat{D}^{\leqslant 0}_{q,(v)}, \hat{D}^{\geqslant 0}_{q,(v)})$ (2.11c) *then real:* $D^b(\hat{\mathscr{G}}_{q,(v)})\to D^b(\mathrm{mod}\,(\Lambda,\Omega))$ *is also an uivalence.*

Proof. This follows from 4.1, 4.2 and the sufficiency part of Theorem 3.3.

5. Some examples

(5.1) Let (Λ, Ω) be the quiver $1 \to 2 \to 3 \to 4$. Consider the data $\{(\alpha_{11}), (\alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}), (\alpha_{31}, \alpha_{32}), (\alpha_{41})\} = \{(4), (3,2,3,1), (2,3), (2)\}$. Let α, β, γ, δ denote the simple roots of the Dynkin diagram $1 \to 2 \to 3 \to 4$ corresponding to the vertices 1, 2, 3, 4. We identify indecomposables of $\mathrm{mod}(\Lambda, \Omega)$ with the corresponding positive roots (via Gabriel's theorem). Identify $\mathrm{mod}(\Lambda_p, \Omega'_p)$, $\mathrm{mod}(\Lambda_p, \Omega_p)$ etc. with the image in $D^b(\mathrm{mod}(\Lambda, \Omega))$ under composite functors: $D^b(\mathrm{mod}(\Lambda_p, \Omega'_p)) \xrightarrow{R'_p} D^b(\mathrm{mod}(\Lambda_p, \Omega_p)) \xrightarrow{j_*} D^b(\mathrm{mod}(\Lambda_{p-1}, \Omega'_{p-1}))$ $\xrightarrow{R'_{p-1}} \cdots \xrightarrow{j_*} \cdots \cdots \xrightarrow{R'_1} D^b(\mathrm{mod}(\Lambda_1, \Omega_1))$. We will indicate the quivers (Λ_p, Ω'_p) {resp. (Λ_p, Ω_p)} by writing the simple objects of $\mathrm{mod}(\Lambda_p, \Omega'_p)$ {resp. (Λ_p, Ω_p)} (for the above imbedding into $D^b(\mathrm{mod}(\Lambda, \Omega))$ in the place of the corresponding vertices.

$(\Lambda_1, \Omega'_1) = (\Lambda_1, \Omega_1) = (\Lambda, \Omega)$, indicated by $\alpha \to \beta \to \gamma \to \delta$,

(Λ_2, Ω_2) $\alpha \to \beta \to \gamma$

$(\Lambda_2, s_{21}\Omega_2)$ $\alpha \to \beta + \gamma \leftarrow \gamma[1]$,

$(\Lambda_2, s_{22}s_{21}\Omega_2)$ $\alpha + \beta + \gamma \leftarrow (\beta + \gamma)[1] \to \beta$,

$(\Lambda_2, s_{23}s_{22}s_{21}\Omega_2) = (\Lambda_2, \Omega'_2)$ $\alpha + \beta + \gamma \leftarrow \gamma[1] \leftarrow \beta[1]$,

(Λ_3, Ω_3) $\gamma[1] \leftarrow \beta[1]$,

$(\Lambda_3, s_{31}\Omega_3) = (\Lambda_3, \Omega'_3)$ $\gamma[2] \to (\beta + \gamma)[1]$,

$(\Lambda_4, \Omega_4) = (\Lambda_4, \Omega'_4)$ $\gamma[2]$.

Next, we describe $(\rho_p \rho_{p+1} \cdots \rho_q)(V(q+1))$ (cf. 3.3a). Recall (3.1a) $\rho_q = j_! R'_{q+1}$ where $j_!: D^b(\mathrm{mod}(\Lambda_{q+1}, \Omega_{q+1})) \to D^b(\mathrm{mod}(\Lambda_q, \Omega'_q))$ (1.8c). The functor $j_!$ is in general easy to write down. Identify $\mathrm{mod}(\Lambda_{q+1}, \Omega_{q+1})$ and $\mathrm{mod}(\Lambda_q, \Omega'_q)$ with subcategories of $D^b(\mathrm{mod}(\Lambda, \Omega))$ as indicated previously. Let $X \in \mathrm{mod}(\Lambda_{q+1}, \Omega_{q+1})$. Then $Y = j_! X$ is the extension of X obtained as follows: Let $\theta_1, \ldots, \theta_k$ be the vertices of $(\Lambda_{q+1}, \Omega_{q+1})$ adjacent to α_{qv_q}. Then $Y(\alpha_{qv_q}) = X(\theta_1) \oplus X(\theta_2) \oplus \cdots \oplus X(\theta_k)$. The morphism $Y(\theta_i \alpha_{qv_q})$ is the inclusion $X(\theta_i) \to Y(\alpha_{qv_q})$. In particular if $X(\theta_i) = 0$ for $i = 1, \ldots, k$ then $Y(\alpha_{qv_q}) = 0$ and $j_! X = j_* X$.

For example, suppose in our example $q = 2$. Identify $\mathrm{mod}(\Lambda_{q+1}, \Omega_{q+1})$ and $\mathrm{mod}(\Lambda_q, \Omega'_q)$ with subcategories of $D^b(\mathrm{mod}(\Lambda, \Omega))$ as indicated previously. Then $j_!(\gamma[1]) = \alpha + \beta$, $j_!(\beta[1]) = \beta[1]$ and $j_!((\beta + \gamma)[1]) = \alpha$.

Coming back to the description of $\rho_p \rho_{p+1} \cdots \rho_q(V(q+1))$, we have

$V(4) = \gamma[2]$

$\{V(3), \rho_3(V(4))\} = \{(\beta + \gamma)[1], \beta[1]\}$

$$\{V(2), \rho_2(\beta+\gamma)[1], \rho_2\beta[1]\} = \{\alpha+\beta+\gamma, \alpha, \beta[1]$$

$$\{V(1), \rho_1(\alpha+\beta+\gamma), \rho_1(\alpha), \rho_1\beta[1]\} = \{\delta, \alpha+\beta+\gamma+\delta, \alpha, \beta[1]\}.$$

The conditions 3.3a are clearly satisfied; for instance $\rho_1\rho_2\rho_3(V(4)) = \beta[1]$ which does not belong to $\mathscr{G}'(1)$ $(= \mathrm{mod}(\Lambda_1, \Omega_1'))$ but $\rho_1\rho_2\rho_3(V(4)) = j_*\rho_2\rho_3(V(4))$.

(5.2) Let (ρ, Ω) be the quiver $1 \to 2 \to 3 \to 4$. Consider the data $\{(4), (3,2,1,3), (2), (1)\}$. We follow the same conventions as in 5.1.

$(\Lambda_1, \Omega_1) = (\Lambda_1, \Omega_1') = (\Lambda, \Omega)$	$\alpha \to \beta \to \gamma \to \delta$,
(Λ_2, Ω_2)	$\alpha \to \beta \to \gamma$,
s_{21}	$\alpha \to \beta+\gamma \leftarrow \gamma[1]$,
s_{22}	$\alpha+\beta+\gamma \leftarrow (\beta+\gamma)[1] \to \beta$
$(\Lambda_2, s_{23}\cdot s_{22}\cdot s_{21}\Omega_2)$ $= (\Lambda_2, \Omega_2')$	$(\alpha+\beta+\gamma)[1] \to \alpha \to \beta$,
$(\Lambda_3, \Omega_2) = (\Lambda_3, \Omega_3')$	$(\alpha+\beta+\gamma)[1] \to \alpha$,
$(\Lambda_4, \Omega_4) = (\Lambda_4, \Omega_4')$	$(\alpha+\beta+\gamma)[1]$,

$\rho_p\rho_{p+1}\cdots\rho_q(V(q+1))$ are as follows.

$$V(4) = (\alpha+\beta+\gamma)[1],$$

$$\{V(3), \rho_3(V(4))\} = \{\alpha, (\beta+\gamma)[1]\},$$

$$\{V(2), \rho_2\alpha, \rho_2(\beta+\gamma)[1]\} = \{\beta, \alpha+\beta, \gamma[1]\},$$

$$\{V(1), \rho_1\beta, \rho_1(\alpha+\beta), \rho_1\gamma[1]\} = \{\delta, \beta, \alpha+\beta, (\gamma+\delta)[1]\}.$$

The condition 3.3a is not satisfied; $\rho_1\rho_2\rho_3 V(4) = \gamma+\delta[1] \notin \mathscr{G}'(1)$ and $j_*R_2'\rho_2\rho_3 V(4) = j_*R_2'\gamma[1] = \gamma[1] \neq \rho_1\rho_2\rho_3 V(4)$.

6. Theorem 7.1 with hypothesis

Let A and B be finite dimensional k-algebras and M an $A-B$ bimodule. Following [B], we say that (A, M, B) is a tilting triple if

(i) $\mathrm{Ext}^1(M, M) = 0$. (6.1)
(ii) $\mathrm{Ext}^i(M, N) = 0$ for $i \geqslant 2$ and $N \in \mathrm{mod}(A)$.
(iii) $\exists$ an exact sequence $0 \to A \to T_1 \to T_2 \to 0$ where T_1 and T_2 are direct summands of finite direct sums of M.
(iv) $B^{\mathrm{opp}} = \mathrm{End}_A(M)$.

One important fact is that $D^b(\mathrm{mod}\, B) \approx D^b(\mathrm{mod}\, A)$ when A is of finite cohomological dimension, [5, § 1]. We review this in a convenient form.

(6.2) Let (A, M, B) be a tilting triple. Let $(D_B^{\leqslant 0}, D_B^{\geqslant 0})$ {resp. $(D_A^{\leqslant 0}, D_A^{\geqslant 0})$} be the natural

t-structure of $D^b(\text{mod } B)$ {resp. $D^b(\text{mod } A)$}. Let $\tilde{D}_B^{\leqslant 0} = \{K \in D_A^{\leqslant 0} | \text{Hom}(M, K[i]) = 0 \forall i > 0\}$. Define $\tilde{D}_B^{\leqslant -1} = \tilde{D}_B^{\leqslant 0}[1]$ and $\tilde{D}_B^{\geqslant 0} = \{L \in D^b(\text{mod } A) | \text{Hom}(K, L) = 0, \forall K \in \tilde{D}_B^{\leqslant -1}\}$. Then $(\tilde{D}_B^{\leqslant 0}, \tilde{D}_B^{\geqslant 0})$ is a t-structure in $D^b(\text{mod } A)$. We have $D_A^{\leqslant -1} \subset \tilde{D}_B^{\leqslant 0} \subset D_A^{\leqslant 0}$. There exists an equivalence $\varphi: D^b(\text{mod } B) \to D^b(\text{mod } A)$ such that $\varphi(D_B^{\leqslant 0}) \subset \tilde{D}_B^{\leqslant 0}$ and $\varphi(D_B^{\geqslant 0}) \subset \tilde{D}_B^{\geqslant 0}$. Furthermore if B is regarded as a module over itself in the canonical way then $\varphi(B) \approx M$. {Throughout we will assume that when M is decomposed into indecomposables (in mod A) the multiplicities are $\leqslant 1$.}

(6.3) Example. In the context of Lemma 2.5a fix a q and assume

$$D_q^{\leqslant -1} \subset R'_q D'^{\leqslant 0}_q.$$

Let $B = A'_q$ be the quiver algebra of (Λ_q, Ω'_q) and $A = A_q$ the quiver algebra of (Λ_q, Ω_q). Let $\varphi = R'_q: D^b(\text{mod } A'_q) \to D^b(\text{mod } A_q)$. Let $M_q = \varphi(A'_q)$. Suppose

$$M_q \in \text{mod}(A_q).$$

Then (A_q, M_q, A'_q) is a tilting triple.

To see this, write $\tilde{D}_B^{\leqslant 0} = \varphi(D'^{\leqslant 0}_q)$ and $\tilde{\mathscr{G}}_B = \varphi(\text{mod}(\Lambda_q, \Omega'_q)$. Then

$$D_q^{\leqslant -1} \subset \tilde{D}_B^{\leqslant 0} \subset D_q^{\leqslant 0} \subset \tilde{D}_B^{\leqslant 1}.$$

Thus, by 3.15a any object X of mod $A = D_q^{\leqslant 0} \cap D_q^{\geqslant 0}$ can be written as $X' \oplus X''$ where $X' \in \tilde{\mathscr{G}}_B[-1]$ and $X'' \in \tilde{\mathscr{G}}_B$. Write $A = A' \oplus A''$ in this fashion. As $A'[1] \in \tilde{\mathscr{G}}_B \approx \text{mod } B (B = A'_q)$, we have a projective resolution $0 \to P'_1 \to P'_2 \to A'[1] \to 0$ in mod B and hence a distinguished triangle $A' \to P'_1 \to P'_2$. Also as A is projective in mod A, $\text{Hom}(A, Y) = 0 \forall Y \in D_q^{\leqslant -1}$. As A'' is a direct summand of A and $\tilde{D}_B^{\leqslant -1} \subset D_q^{\leqslant -1}$, $\text{Hom}(A'', Y) = 0 \forall Y \in \tilde{D}_B^{\leqslant -1}$. From this we conclude that A'' is projective in mod B. Thus, we have a distinguished triangle.

$$A' \oplus A'' \to P_1 \oplus A'' \to P_2, \text{ i.e., } A \to P_1 \oplus A'' \to P_2. \tag{6.3a}$$

But the projectives of $\tilde{\mathscr{G}}_B (\approx \text{mod } B = \text{mod}(\Lambda_q, \Omega'_q) = \text{mod}(A'_q))$ are direct summands of direct sums of $M \approx \varphi(B)$. Thus 6.3a gives rise to an exact sequence $0 \to A \to T_1 \to T_2 \to 0$ of the type required in 6.1 (iii). The other conditions in 6.1 are easy to verify.

(6.4) For $1 \leqslant p \leqslant q \leqslant n$, define by descending induction on p a t-structure $(\tilde{D}_{q,p}^{\leqslant 0}, \tilde{D}_{q,p}^{\geqslant 0})$ in $D^b(\text{mod}(\Lambda_p, \Omega'_p))$ and $D^b(\text{mod}(\Lambda_p, \Omega_p))$ by

$$\tilde{D}_{q,p}^{\leqslant 0} = \{X \in D_p'^{\leqslant 0} | j^* X \in \tilde{D}_{q,p+1}^{\leqslant 0}\}.$$

$\tilde{D}_{q,p}^{\leqslant 0}$ in $D^b(\text{mod}(\Lambda_p, \Omega_p))$ is obtained by pushing the above $\tilde{D}_{q,p}^{\leqslant 0}$ by the isomorphism R'_p. Here $j^*: D^b(\text{mod}(\Lambda_p, \Omega'_p)) \to D^b(\text{mod}(\Lambda_{p+1}, \Omega_{p+1}))$ is the functor 1.9. Also to start induction, we define $(\tilde{D}_{q,q}^{\leqslant 0}, \tilde{D}_{q,q}^{\geqslant 0})$ to be $(D'^{\leqslant 0}_q, D'^{\geqslant 0}_q)$, the natural t-structure of $D^b(\text{mod}(\Lambda_q, \Omega'_q))$ and its pushforward by R'_q in $D^b(\text{mod}(\Lambda_q, \Omega_q))$.

(6.4a) We denote the heart of $(\tilde{D}_{q,p}^{\leqslant 0}, \tilde{D}_{q,p}^{\geqslant 0})$ by $\tilde{\mathscr{G}}_{q,p}$. Observe the following relation between the t-structures 2.11c and the above:

$$(\hat{D}_{q,(v_q-1)}^{\leqslant 0}, \hat{D}_{q,(v_q-1)}^{\geqslant 0}) = (\tilde{D}_{q,1}^{\leqslant 0}, \tilde{D}_{q,1}^{\geqslant 0}).$$

As a first step in proving Theorem 7.1, we will show

PROPOSITION 6.5

Given a data 2.1, assume that

(i) *real:* $D^b(\tilde{\mathscr{G}}_1) \to D^b(\mathrm{mod}(\Lambda, \Omega))$ *is an isomorphism*

(ii) (ii) *the assumption in 6.3 is satisfied for all* q, $1 < q \leqslant n$, *i.e.,* $D_q^{\leqslant -1} \subset R'_q D'^{\leqslant 0}_q$ *and* $R'_q(A'_q) \in \mathrm{mod}(\Lambda_q, \Omega_q)$ *(cf. 6.3), in other words* $R'_q(P') \in \mathrm{mod}(\Lambda_q, \Omega_q)$ *for each indecomposable projective* P' *of* $\mathrm{mod}(\Lambda_q, \Omega'_q)$.

Then for $1 \leqslant q \leqslant n$ $\exists$ *a finite dimensional k-algebra* $\tilde{A}_{q,1}$ *such that* $\mathrm{mod}\,\tilde{A}_{q,1} \approx \tilde{\mathscr{G}}_{q,1}$ ($\tilde{\mathscr{G}}_{q,p}$ *is the heart of the t-structure* $(\tilde{D}^{\leqslant 0}_{q,p}, \tilde{D}^{\geqslant 0}_{q,p})$.) *Furthermore, there exists an* $\tilde{A}_{q,1} - \tilde{A}_{q+1,1}$ *bimodule* $\tilde{M}_{q+1,1}$ *such that* $(\tilde{A}_{q,1}, \tilde{M}_{q+1,1}, \tilde{A}_{q+1,1})$ *is a tilting triple.*

As a technical step required in the inductive proof of 6.5, we will in fact show

PROPOSITION 6.6

Under the same hypotheses as in 6.5, $\exists$ *finite dimensional k-algebras* $\tilde{A}_{q,p}$ *such that* $\mathrm{mod}(\tilde{A}_{q,p}) \approx \tilde{\mathscr{G}}_{q,p}$. *There exist* $\tilde{A}_{q,p} - \tilde{A}_{q+1,p}$ *bimodules* $\tilde{M}_{q+1,p}$ *such that* $(\tilde{A}_{q,p}, \tilde{M}_{q+1,p}, \tilde{A}_{q+1,p})$ *is a tilting triple.*

Proof. Let M_{p+1} be the tilting module of $\mathrm{mod}(A_{p+1})$ described in 6.3 giving rise to the tilting triple $(A_{p+1}, M_{p+1}, A'_{p+1})$. Thus if we identify $\mathrm{mod}(\Lambda_{p+1}, \Omega'_{p+1})$ as a subcategory of $D^b(\mathrm{mod}(\Lambda_{p+1}, \Omega_{p+1}))$ using the isomorphism R'_{p+1}, then M_{p+1} is the direct sum of the projective indecomposables of $\mathrm{mod}(\Lambda_{p+1}, \Omega'_{p+1})$ (one copy each). Let $V(p)$ be the simple object of $\mathrm{mod}(\Lambda_p, \Omega'_p)$ corresponding to the sink α_{p,ν_p}. If $j_!$ is the usual functor $D^b(\mathrm{mod}(\Lambda_{p+1}, \Omega_{p+1})) \to D^b(\mathrm{mod}(\Lambda_p, \Omega'_p))$, set $\tilde{M}_{p+1,p} = j_!(M_{p+1}) \oplus V(p)$.

(6.7) We claim that $\tilde{M}_{p+1,p}$ is a tilting module in $\mathrm{mod}(A'_p)$ ($\mathrm{mod}\, A'_p \approx \tilde{\mathscr{G}}_{p,p}$).
More generally, define

$$\begin{aligned} \tilde{M}_{q+1,p} = {} & \rho_p \cdot \rho_{p+1} \cdots \rho_q (A'_{q+1}) \\ & \oplus \rho_p \cdot \rho_{p+1} \cdots \rho_{q-1}(V(q)) \\ & \oplus \rho_p \cdots \rho_{q-2}(V(q-1)) \\ & \oplus \cdots \\ & \oplus V(p) \end{aligned} \tag{6.8}$$

and $\tilde{A}_{q,p} = \mathrm{End}(\tilde{M}_{q,p})^{\mathrm{opp}}$.

(6.9) Assume inductively on q that $\mathrm{mod}\,\tilde{A}_{l,p} \approx \tilde{\mathscr{G}}_{l,p}$ the heart of $(\tilde{D}^{\leqslant 0}_{l,p}, \tilde{D}^{\geqslant 0}_{l,p})$ for $1 \leqslant l \leqslant q$ (and all p such that $1 \leqslant p \leqslant l$). Assume further that $(\tilde{A}_{l,p}, \tilde{M}_{l+1,p}, \tilde{A}_{l+1,p})$ is a tilting triple for $1 \leqslant l < q$ and $1 \leqslant p \leqslant l$ and also that

$$\tilde{D}^{\leqslant 0}_{l+1,p} = \{K \in \tilde{D}^{\leqslant 0}_{l,p} \mid \mathrm{Hom}(\tilde{M}_{l+1,p}, K[i]) = 0, \forall i > 0\} \tag{6.9a}$$

(For motivation compare the above hypotheses with 6.2)

(6.10) We wish to assert the validity of 6.9 and 6.9a replacing q by $q+1$.
For this we have to show that

$$\operatorname{mod} \tilde{A}_{q+1,p} \approx \tilde{\mathscr{G}}_{q+1,p} \quad \text{for} \quad 1 \leqslant p \leqslant q+1; \tag{6.11}$$

$(\tilde{A}_{q,p}, \tilde{M}_{q+1,p}, \tilde{A}_{q+1,p})$ is a tilting triple and

$$\tilde{D}^{\leqslant 0}_{q+1,p} = \{K \in \tilde{D}^{\leqslant 0}_{q,p} \mid \operatorname{Hom}(\tilde{M}_{q+1,p}, K[i]) = 0, \forall i > 0\}. \tag{6.11a}$$

First we show $\tilde{M}_{q+1,p} \in \tilde{\mathscr{G}}_{q,p}$ $(\approx \operatorname{mod} \tilde{A}_{q,p})$.
Let Y be a direct summand of $\tilde{M}_{q+1,p}$. Then

(6.12)

(a) either $Y = V(p)$ or
(b) $Y = j_! R'_{p+1}(Z)$ where Z is a direct summand of $\tilde{M}_{q+1,p+1}$ and $j_! : D^b(\operatorname{mod}(\Lambda_{p+1}, \Omega_{p+1})) \to D^b(\operatorname{mod}(\Lambda_p, \Omega'_p))$ is the usual functor.

(This follows from 6.8). In the case 6.12(a) trivially $Y \in \tilde{\mathscr{G}}_{q,p}$.

(6.13) Consider the case 6.12b; i.e., $Y = j_! R'_{p+1}(Z)$. The data 2.1 gives rise to a data $\{\alpha_{t1}, \ldots, \alpha_{tv_t} \mid p \leqslant t \leqslant n\}$ in (Λ_p, Ω_p), for which $(\tilde{D}^{\leqslant 0}_{l,p}, \tilde{D}^{\geqslant 0}_{l,p})$ is one of the intermediate t-structures 2.11c (cf. 6.4a). By 3.10 and 4.4 we conclude

$$\text{real: } D^b(\tilde{\mathscr{G}}_{l,p}) \to D^b(\operatorname{mod}(\Lambda_p, \Omega_p) \text{ is an equivalence.} \tag{6.13a}$$

As a consequence Theorem 3.3 can be applied. The result of this application, one checks, is that

(i) either $j_! R'_{p+1}(Z) \in \mathscr{G}'(p)$ or (6.14)

(ii) $j_! R'_{p+1}(Z) = j_* R'_{p+1}(Z)$.

In the latter case, an obvious descending induction on p (namely, $\tilde{M}_{q+1,p+1} \in \tilde{\mathscr{G}}_{q,p+1}$) and the fact that j_* is exact with respect to the t-structures $(\tilde{D}^{\leqslant 0}_{q,p+1}, \tilde{D}^{\geqslant 0}_{q,p+1})$ and $(\tilde{D}^{\leqslant 0}_{q,p}, \tilde{D}^{\geqslant 0}_{q,p})$ show that $Y \in \tilde{\mathscr{G}}_{q,p}$. Starting the induction is taken care of by the assumption 6.5 (ii).

In the case 6.14 (i), we have $j_! R'_{p+1}(Z) \in \mathscr{G}'(p) \subset D'^{\geqslant 0}_p \subset \tilde{D}^{\geqslant 0}_{q,p}$. But also $j_! R'_{p+1}(Z) \in \tilde{D}^{\leqslant 0}_{q,p}$ as $j_!$ is right exact [1, 1.4.16] and $R'_{p+1}(Z) \in \tilde{D}^{\leqslant 0}_{q,p+1}$. Thus $j_! R'_{p+1}(Z) \in \tilde{D}^{\leqslant 0}_{q,p} \cap \tilde{D}^{\geqslant 0}_{q,p} = \tilde{\mathscr{G}}_{q,p}$.

This completes the proof that $\tilde{M}_{q+1,p} \in \tilde{\mathscr{G}}_{q,p}$. Next, we show that

$$\operatorname{Ext}^i_{\tilde{A}_{q,p}}(\tilde{M}_{q+1,p}, X) = 0 \quad \text{for} \quad i \geqslant 2 \text{ and any } X \in \operatorname{mod}(\tilde{A}_{q,p}). \tag{6.15}$$

But $\tilde{M}_{q+1,p} = V(p) \oplus j_! R'_{p+1}(\tilde{M}_{q+1,p+1})$ (6.8). As real: $D^b(\tilde{\mathscr{G}}_{q,p}) \to D^b(\operatorname{mod}(\Lambda_p, \Omega_p)$ is an equivalence (6.13a), $\operatorname{Ext}^i(\tilde{M}_{q+1,p}, X) = \operatorname{Hom}(\tilde{M}_{q+1,p}, X[i])$ in the derived category. Since $\tilde{\mathscr{G}}_{q,p} \subseteq \tilde{D}^{\leqslant 0}_{q,p} \subseteq D'^{\leqslant 0}_p$ and as $V(p)$ is a simple projective object of $\mathscr{G}'(p)$, $\operatorname{Hom}(V(p), X[i]) = 0$ for $X \in \tilde{\mathscr{G}}_{q,p}$ and $i > 0$. $\operatorname{Hom}(j_! R'_{p+1}(\tilde{M}_{q+1,p+1}), X[i]) = \operatorname{Hom}(R'_{p+1}(\tilde{M}_{q+1,p+1}), j^* X[i])$. But $j^* X \in \tilde{\mathscr{G}}_{q,p+1}$. This allows us to conclude $\operatorname{Hom}(j_! R'_{p+1}(\tilde{M}_{q+1,p+1}), X[i]) = 0$, for $i > 1$ by using the induction hypothesis $\operatorname{Hom}(\tilde{M}_{q+1,p+1}, Y[i]) = 0$ for $i > 1$ and $Y \in \tilde{\mathscr{G}}_{q,p+1}$. Starting the induction is taken care

of by the assumption 6.5 (ii): $\mathrm{Hom}(R'_q(A'_q), Z) = 0$ for $Z \in R'_q(D'^{\leqslant -1}_q)$, hence for $Z \in D^{\leqslant -2}_q$ as $D^{\leqslant -2}_q \subset R'_q D'^{\leqslant -1}_q$ (6.5 ii).

This completes the proof of 6.15. Next we will show

$$\mathrm{Ext}^1_{\tilde{A}_{q,p}}(\tilde{M}_{q+1,p}, \tilde{M}_{q+1,p}) = 0. \tag{6.16}$$

Again, this ext group is the same as $\mathrm{Hom}(\tilde{M}_{q+1,p}, \tilde{M}_{q+1,p}[1]) = \mathrm{Hom}(V(p) \oplus j_! R'_{p+1}(\tilde{M}_{q+1,p+1}), V(p)[1] \oplus j_! R'_{p+1}(\tilde{M}_{q+1,p+1})[1])$. As already seen during the proof 6.15, $\mathrm{Hom}(V(p), \tilde{M}_{q+1,p}[1]) = 0$. Also, if we assume inductively $\mathrm{Hom}(\tilde{M}_{q+1,p+1}, \tilde{M}_{q+1,p+1}[1]) = 0$, then $\mathrm{Hom}(j_! R'_{p+1}(\tilde{M}_{q+1,p+1}), j_! R'_{p+1}(\tilde{M}_{q+1,p+1})[1]) = 0$. Thus we are reduced to showing $\mathrm{Hom}(j_! R'_{p+1}(\tilde{M}_{q+1,p+1}), V(p)[1]) = 0$. The latter equals $\mathrm{Hom}(R'_{p+1}(\tilde{M}_{q+1,p+1}), j^* V(p)[1]) = 0$, as $j^* V(p) = 0$.

This proves 6.16.

The expression 6.8 allows us to count the number of indecomposable direct summands of $\tilde{M}_{q+1,p}$. By a result of Bongartz [2], we can replace the condition 6.1 (iii) in the definition of a tilting module by

(6.17) The number of nonisomorphic indecomposable summands in M as an A-module equals the number of distinct simple A-modules.

Thanks to this result we can now conclude $(\tilde{A}_{q,p}, \tilde{M}_{q+1,p}, \tilde{A}_{q+1,p})$ is a tilting triple. Only 6.11a remains to be shown. The assertion $\mathrm{mod}\, \tilde{A}_{q+1,p} \approx \tilde{\mathscr{G}}_{q+1,p}$ in 6.11 is then deduced from the isomorphism φ in 6.2.

Proof of 6.11a. Let $K \in$ right side of 6.11a. Then in particular

$$\mathrm{Hom}(j_! R'_{p+1} \tilde{M}_{q+1,p+1}, K[i]) = 0, \quad \forall i > 0. \tag{6.18}$$

But $\mathrm{Hom}(j_! R'_{p+1} \tilde{M}_{q+1,p+1}, K[i]) \approx \mathrm{Hom}(R'_{p+1} \tilde{M}_{q+1,p+1}, j^* K[i])$. So

$$\mathrm{Hom}(R'_{p+1} \tilde{M}_{q+1,p+1}, j^* K[i]) = 0 \quad \forall i > 0. \tag{6.19}$$

Make a (descending) induction hypothesis

$$\tilde{D}^{\leqslant 0}_{q+1,p+1} = \{K' \in \tilde{D}^{\leqslant 0}_{q,p+1} \mid \mathrm{Hom}(\tilde{M}_{q+1,p+1}, K'[i]) = 0, \forall i > 0\}. \tag{6.20}$$

By 6.19, $j^* K \in R'_{p+1}(\tilde{D}^{\leqslant 0}_{q+1,p+1})$. Now $6.4 \Rightarrow j_! j^* K \in \tilde{D}^{\leqslant 0}_{q+1,p} \subset D'^{\leqslant 0}_p$. Hence (6.21)

$$\mathrm{Hom}(V(p), j_! j^* K[i]) = 0 \quad \text{for } i > 0. \tag{6.22}$$

Applying $\mathrm{Hom}(V(p), *)$ to the triangle $j_! j^*(K[i]) \to K[i] \to i_* i^*(K[i]) \to j_! j^*(K[i+1])$, $6.22 \Rightarrow \mathrm{Hom}(V(p), K[i]) \approx \mathrm{Hom}(V(p), i_* i^* K[i])$. By choice of K ($\in$ right side of 6.11a), $\mathrm{Hom}(V(p), K[i]) = 0$ for $i > 0$. So, $\mathrm{Hom}(V(p), i_* i^* K[i]) = 0$ for $i > 0$. Therefore (a trivial property about the natural t-structure $(D^{\leqslant 0}_F, D^{\geqslant 0}_F)$ in $D^b(\mathrm{mod}\, k)$)

$$i^* K \in D^{\leqslant 0}_F \quad \text{and} \quad i_* i^* K \in \tilde{D}^{\leqslant 0}_{q+1,p}. \tag{6.23}$$

Now in the triangle $j_! j^* K \to K \to i_* i^* K$ both $j_! j^* K$ and $i_* i^* K \in \tilde{D}^{\leqslant 0}_{q+1,p}$. So $K \in \tilde{D}^{\leqslant 0}_{q+1,p}$. This completes the proof of one implication in 6.11a. For the other implication

Suppose $K \in \tilde{D}^{\leqslant 0}_{q+1,p}$, so that $j^* K \in \tilde{D}^{\leqslant 0}_{q+1,p+1}$ (cf. 6.4). We have to show that

$\mathrm{Hom}(\tilde{M}_{q+1,\underline{p}}, K[i]) = 0, \forall i > 0$. As $\tilde{D}^{\leqslant 0}_{q+1,p} \subset D'^{\leqslant 0}_p$ $\mathrm{Hom}(V(p), K[i]) = 0, \forall i > 0$. Recall

(6.8) $\tilde{M}_{q+1,p} = V(p) \oplus j_! R'_{p+1}(\tilde{M}_{q+1,p+1})$. $\mathrm{Hom}(j_! R'_{p+1}(\tilde{M}_{q+1,p+1}), K[i])$ $= \mathrm{Hom}(R'_{p+1}(\tilde{M}_{q+1,p+1}), j^* K[i] = 0$ for $i > 0$ as $j^* K \in \tilde{D}^{\leqslant 0}_{q+1,p+1}$ and we have 6.20.

This completes the proof of 6.6 and 6.5.

7. Removing hypothesis of section 6

(7.1) **Theorem.** *Suppose given a data* (2.1) *with associated t-structure* (2.9) $(\tilde{D}^{\leqslant 0}_1, \tilde{D}^{\geqslant 0}_1)$. *Assume real:* $D^b(\tilde{\mathscr{G}}_1) \to D^b(\mathrm{mod}(\Lambda, \Omega))$ *is an equivalence. Then the data* $\{\alpha_{p1}, \ldots, \alpha_{pv_p} | 1 \leqslant p \leqslant n\}$ *can be modified to another data* $\{\beta_{p1}, \ldots, \beta_{p\mu_p} | 1 \leqslant p \leqslant n\}$ *such that*

(a) *both data have the same associated t-structure*

$$(\tilde{D}^{\leqslant 0}_1, \tilde{D}^{\geqslant 0}_1).$$

(b) *the data* $\{\beta_{p1}, \ldots, \beta_{p\mu_p} | 1 \leqslant p \leqslant n\}$ *satisfies the conditions assumed in Proposition 6.5.*

In particular there exist finite dimensional k-algebras $A(1), \ldots, A(n)$ *such that* (i) $A(j-1)$ *is obtained by tilting* $A(j)$, (ii) $\mathrm{mod}(A_1) \approx \tilde{\mathscr{G}}_1$ *and* (iii) $A(n)$ *is the quiver algebra of* $(\Lambda, \bar{\Omega})$ (*a quiver without relations*) *where* $(\Lambda, \bar{\Omega})$ *and* (Λ, Ω) *have the same underlying graph but possibly with different orientations.*

Proof. Fix p, $1 < p \leqslant n$. Suppose that the given data $\{\alpha_{q1}, \ldots, \alpha_{qv_q} | 1 \leqslant q \leqslant n\}$ has been modified to another data $\{\alpha'_{q1}, \ldots, \alpha'_{qv'_q} | 1 \leqslant q \leqslant n\}$ such that

(7.2)

(i) both data have the same associated t-structure (2.9).
(ii) the condition (ii) in 6.5 is satisfied for the $\alpha'_{\cdot\cdot}$ data for $p+1 \leqslant q \leqslant n$, i.e., $D^{\leqslant -1}_q \subseteq R'_q D'^{\leqslant 0}_q$ and $R'_q(A'_q) \in \mathscr{G}(q)$ for $p+1 \leqslant q \leqslant n$ where the notations refer to objects associated to the $\alpha'_{\cdot\cdot}$ data.

We will then show that it is possible to modify $\{\alpha'_{q1}, \ldots, \alpha'_{qv'_q} | 1 \leqslant q \leqslant n\}$ to a data $\{\alpha''_{q1}, \ldots, \alpha''_{qv''_q} | 1 \leqslant q \leqslant n\}$ such that

(7.3)

(i) both data have the same associated t-structure.
(ii) the condition (ii) in 6.5 is satisfied for the α'' data for $p \leqslant q \leqslant n$.

Obviously, we can take the data $\alpha''_{\cdot\cdot}$ = data $\alpha'_{\cdot\cdot}$ if for the $\alpha'_{\cdot\cdot}$ data

(7.4)

(a) $D^{\leqslant -1}_p \subseteq D'^{\leqslant 0}_p$ and
(b) any projective of $\mathscr{G}'(p)$ belongs to $\mathscr{G}(p)$

(in other words, if the condition 7.2 iii is satisfied for $q = p$). Until the $\alpha''_{\cdot\cdot}$ data is constructed, in the following notation the objects are associated to $\alpha'_{\cdot\cdot}$ data.

If 7.4(a) is not satisfied consider the smallest v, $1 \leqslant v \leqslant v'_p$ such that $D^{\leqslant -1}_p \subset \tilde{D}^{\leqslant 0}_{p,(v)}$ and $D^{\leqslant -1}_p \not\subset \tilde{D}^{\leqslant 0}_{p,(v+1)}$ where $\tilde{D}^{\leqslant 0}_{\cdot\cdot}$ are defined following 2.11(b). The sink $\alpha'_{p,v+1}$ of

$(\Lambda_p, s_{p,\nu}\cdots s_{p,1}\Omega_p)$ gives rise to a simple projective P of $\tilde{\mathscr{G}}_{p,(\nu)}$. By the minimality of ν and 2.12, one infers that

$$P[-1]\in\mathscr{G}(p), \quad \text{i.e.,} \quad P\in\mathscr{G}(p)[1]. \tag{7.5}$$

Let $j_!j_*: D^b(\mathrm{mod}\,(\Lambda_p, \Omega_p))\to D^b(\mathrm{mod}\,(\Lambda_{p-1}, \Omega_{p-1}))$ be the usual functors. By 4.2, either $j_!(P)=j_*(P)$ or $j_!(P)\in\mathscr{G}'(p-1)$. The latter is impossible as it would imply $P(\approx j^*j_!(P))\in\mathscr{G}(p)$ contradicting 7.5. Thus

$$j_!P=j_*P. \tag{7.6}$$

The relation $D_p^{\leqslant -1}\subset \tilde{D}^{\leqslant 0}_{p,(\nu)}$ and the fact P is projective in $\tilde{\mathscr{G}}_{p,(\nu)}\Rightarrow \mathrm{Hom}\,(P, X)=0$ for $X\in D_p^{\leqslant -2}$. Hence $\mathrm{Hom}\,(P[-1], Y)=0$ for $Y\in D_p^{\leqslant -1}$. Together with 7.5, we now conclude that

$$P[-1] \text{ is projective in } \mathscr{G}(p). \tag{7.7}$$

View an object of $\mathscr{G}(p)$ ($\approx \mathrm{mod}\,(\Lambda_p, \Omega_p)$) as a collection of vector spaces over vertices of Λ_p and linear maps corresponding to arrows. Like any nonzero projective of $\mathscr{G}(p)$, the vector space corresponding to $P[-1]$ is nonzero for at least one sink γ of (Λ_p, Ω_p). The relation 7.6 implies that there is no arrow from γ to $\alpha'_{p-1,\nu'_{p-1}}$ in $(\Lambda_{p-1}, \Omega'_{p-1})$ (e.g., use description of $j_!$ in 5.1). Thus, γ is a sink of $(\Lambda_{p-1}, \Omega'_{p-1})$ different from $\alpha'_{p-1,\nu'_{p-1}}$.

(7.8) Let V_γ be the simple object of $\mathscr{G}(p)$ corresponding to the vertex γ of (Λ_p, Ω_p). Since $\mathrm{Hom}\,(V_\gamma, P[-1])\neq 0$ and $P\in\mathrm{mod}\,(\Lambda_p, s_{p\nu}\cdots s_{p1}\Omega_p)$

$$V_\gamma\in\mathrm{mod}\,(\Lambda_p, s_{p\nu}\cdots s_{p1}\Omega_p)[l] \text{ where } l\leqslant -1. \tag{7.9}$$

(This follows using property 1.16a (i) in the definition of a t-structure.)

(7.10) Claim. For $1\leqslant\mu<\nu'$, V_γ is a simple projective of $\tilde{\mathscr{G}}_{p,(\mu)}$. $\mathrm{Hom}\,(V_\gamma, X)=0$ for $X\in D_p^{\leqslant -1}$ as V_γ is projective in $\mathrm{mod}\,(\Lambda_p, \Omega_p)$. Also, $\mathrm{Hom}\,(X, V_\gamma)=0$ for $X\in D_p^{\leqslant -1}$ (1.16a (i)). Since $\tilde{D}^{\leqslant 0}_{p,(\mu)}\subset D_p^{\leqslant 0}$, we have for $X\in\tilde{D}^{\leqslant -1}_{p,(\mu)}$

$$\mathrm{Hom}\,(V_\gamma, X)=0 \quad \text{and} \quad \mathrm{Hom}\,(X, V_\gamma)=0. \tag{7.10a}$$

The last fact $\Rightarrow V_\gamma\in\tilde{D}^{\leqslant 0}_{p,(\mu)}$. Hence $V_\gamma\in\tilde{\mathscr{G}}_{p,(\mu)}$. Then 7.10(a)$\Rightarrow V_\gamma$ is projective in $\tilde{\mathscr{G}}_{p,(\mu)}$. It remains to show that V_γ is a simple object of $\tilde{\mathscr{G}}_{p,(\mu)}$. Since V_γ is a simple projective of $\mathscr{G}(p)$, if $Y\in D_p^{\leqslant 0}$ and $f: Y\to V_\gamma$ is a nonzero morphism then f splits. As $\tilde{\mathscr{G}}_{p,(\mu)}\subset D_p^{\leqslant 0}$, the above remark implies that if $Y\in\tilde{\mathscr{G}}_{p,(\mu)}$ and Y is a nonzero subobject of V_γ in $\tilde{\mathscr{G}}_{p,(\mu)}$ then $Y=V_\gamma$. This completes the proof of 7.10.

It is now easy to conclude that $\gamma\neq\alpha'_{p,\mu}$ $(1\leqslant\mu<\nu')$ and $\gamma=\alpha'_{p,\nu'}$. This property together with the fact that γ is a sink of $(\Lambda_{p-1}, \Omega_{p-1})$ and $\gamma\neq\alpha'_{p-1,\nu'_{p-1}}$ is all we need to construct a new data $\alpha''_{\cdot,\cdot}$ without changing the associated t-structure 2.9.

Define $\nu''_l=\nu'_l$ for $1\leqslant l\leqslant n$, $l\neq p-1, p$, $\nu''_{p-1}=\nu'_{p-1}+1$ and $\nu''_p=\nu'_p-1$. $\alpha''_{l,\nu}=\alpha'_{l,\nu}$ for $l\neq p-1, p$ and $1\leqslant\nu\leqslant\nu'_l$. Define the $\alpha''_{p-1,*}$ sequence by adjoining γ just before $\alpha'_{p-1,\nu'_{p-1}}$ in the $\alpha'_{p-1,*}$ sequence. Define the $\alpha''_{p,*}$ sequence by dropping $\alpha'_{p,\nu'}(=\gamma)$ from the $\alpha'_{p,*}$ sequence. In other words, $\alpha''_{p-1,l}=\alpha'_{p-1,l}$ for $1\leqslant l<\nu'_{p-1}$, $\alpha''_{p-1,\nu'_{p-1}}=\gamma$ and $\alpha''_{p-1,1+\nu'_{p-1}}=\alpha'_{p-1,\nu'_{p-1}}$ while, $\alpha''_{p,l}=\alpha'_{p,l}$ for $1\leqslant l<\nu'$, $\alpha''_{p,l}=\alpha'_{p,1+l}$ for $\nu'\leqslant l\leqslant\nu'_p-1$.

(Λ_q, Ω_q) is the same for both data for $1\leqslant q\leqslant n$, $q\neq p$. (Λ_q, Ω'_q) is the same for both data for

$1 \leqslant q \leqslant n, q \neq p-1$. (Λ_p, Ω_p) for the $\alpha''_{\cdot\cdot}$ data is obtained by reflecting with respect to the sink γ of (Λ_p, Ω_p) for the $\alpha'_{\cdot\cdot}$ data. We identify $D^b(\mathrm{mod}\,(\Lambda_p, \Omega_p))$ for the $\alpha''_{\cdot\cdot}$ data with $D^b(\mathrm{mod}\,(\Lambda_p, \Omega_p))$ for the $\alpha'_{\cdot\cdot}$ data using the reflection functor at γ. Similar remarks apply to $(\Lambda_{p-1}, \Omega'_{p-1})$ and $D^b(\mathrm{mod}\,(\Lambda_{p-1}, \Omega'_{p-1}))$. With these identifications the t-structures $(\tilde{D}_q^{\leqslant 0}, \tilde{D}_q^{\geqslant 0})$ in the inductive construction of Proposition 2.3 coincide for both data for all q, $1 \leqslant q \leqslant n$.

Recall that we are trying to modify the $\alpha'_{\cdot\cdot}$ data to make it satisfy 7.4a (forgetting 7.4b for a while) in addition to 7.2 (i) and (ii).

Note that the $\alpha''_{\cdot\cdot}$ data constructed above satisfies 7.2 (i) and (ii). If it still does not satisfy 7.4(a) we repeat all the steps following 7.4 starting with this $\alpha''_{\cdot\cdot}$ data in place of the $\alpha'_{\cdot\cdot}$ data. At each stage of this iteration (as long as 7.4a is not satisfied) the length of the α''_{p*} sequence falls by one. So eventually after a finite number of steps 7.4a and 7.2 (i), (ii) are all satisfied.

Without loss of generality (or by changing notation) we can now assume that the $\alpha'_{\cdot\cdot}$ data satisfies 7.4a and 7.2 (i), (ii).

Now suppose 7.4(b) is not satisfied. Let P be an indecomposable projective of $\mathscr{G}'(p)$ and suppose $P \notin \mathscr{G}(p)$ so that $P \in D_p^{\leqslant -1}$. Because of 7.4a, $\mathrm{Hom}(P, X) = 0$ if $X \in D_p^{\leqslant -2}$. Thus $P[-1] \in \mathscr{G}(p)$ and is a projective of $\mathscr{G}(p)$. Repeating the steps after 7.5, we produce another data which satisfies 7.2 (i), (ii) and for which the length of the $\alpha''_{p,*}$ sequence is one less than the length of the $\alpha'_{p,*}$ sequence. (This new $\alpha''_{\cdot\cdot}$ data also satisfies 7.4a, but we don't have to prove it; if it doesn't satisfy 7.4a, iterate the steps following 7.4a, b.)

Clearly, by iterating these steps, eventually we arrive at a data satisfying 7.3 (i) and (ii). This completes the proof of Theorem 7.1. q.e.d.

8. Completeness in the case of a Dynkin quiver

In this section, we show that all non-degenerate t-structures (see definition below) arise as the t-structure associated to a data if (Λ, Ω) is a Dynkin quiver. Theorem 2 then enables us to reprove a result of Happel.

We need some preparation, for which the assumption that (Λ, Ω) is a Dynkin quiver is not needed. Recall the functors $i_*, j_*, \ldots$ etc. Also recall that $(D_F^{\leqslant 0}, D_F^{\geqslant 0})$ is the natural t-structure of $D^b(\mathrm{mod}\,k)$. A t-structure $(\tilde{D}^{\leqslant 0}, \tilde{D}^{\geqslant 0})$ is said to be *non-degenerate* if $\bigcap_m D^{\leqslant m}$ and $\bigcap_m D^{\geqslant m}$. (cf. [1, 1.3.7]) both consist of the zero objects.

Lemma 8.1. Let α be a sink of (Λ, Ω). Define (Λ_U, Ω_U) as in 1.8. Let $(D^{\leqslant 0}, D^{\geqslant 0})$ be the natural t-structure of $D^b(\mathrm{mod}\,(\Lambda, \Omega))$. {Thus, in particular, $i_*(D_F^{\leqslant 0}) \subset D^{\leqslant 0}$ and $i_*(D_F^{\geqslant 0}) \subset D^{\geqslant 0}$.} Now, let $(\tilde{D}^{\leqslant 0}, \tilde{D}^{\geqslant 0})$ be any t-structure of $D^b(\mathrm{mod}\,(\Lambda, \Omega))$ such that (i) $\tilde{D}^{\leqslant 0} \subset D^{\leqslant 0}$ and (ii) $i_*(D_F^{\leqslant 0}) \subset \tilde{D}^{\leqslant 0}$. Then there is a unique t-structure $(\tilde{D}_U^{\leqslant 0}, \tilde{D}_U^{\geqslant 0})$ in $D^b(\mathrm{mod}\,(\Lambda_U, \Omega_U))$ such that $\tilde{D}_U^{\leqslant 0} \subset D_U^{\leqslant 0}$ $\{(D_U^{\leqslant 0}, D_U^{\geqslant 0})$ is the natural t-structure of $D^b(\mathrm{mod}\,(\Lambda_U, \Omega_U))\}$ and $\tilde{D}^{\leqslant 0} = \{K \in D^{\leqslant 0} | j^*K \in \tilde{D}_U^{\leqslant 0}\}$. $\tilde{D}_U^{\leqslant 0}$ is given by $\tilde{D}_U^{\leqslant 0} = \{X \in D^b(\mathrm{mod}\,(\Lambda_U, \Omega_U)) | j_*X \in \tilde{D}^{\leqslant 0}\}$.

Proof. Let $\tilde{\tau}_{\leqslant 0}(K) \to K \to \tilde{\tau}_{\geqslant 1}(K)$ denote the truncation triangle of any $K \in D^b(\mathrm{mod}\,(\Lambda, \Omega))$ with respect to the t-structure $(\tilde{D}^{\leqslant 0}, \tilde{D}^{\geqslant 0})$. $(\tilde{\tau}_{\leqslant 0}(K) \in \tilde{D}^{\leqslant 0}$ and $\tilde{\tau}_{\geqslant 1}(K) \in \tilde{D}^{\geqslant 1}$.

(8.2) Let $X \in D^b(\mathrm{mod}\,(\Lambda_U, \Omega_U))$. Let $K = j_*X$. Then $\tilde{\tau}_{\leqslant 0}(K) \approx j_* j^*(\tilde{\tau}_{\leqslant 0}(K))$ and $\tilde{\tau}_{\geqslant 1}(K) \approx j_* j^*(\tilde{\tau}_{\geqslant 1}(K))$.

The morphism $\tilde{\tau}_{\leqslant 0}(K) \to K$ induces isomorphism $\mathrm{Hom}(L, \tilde{\tau}_{\leqslant 0}(K)) \to \mathrm{Hom}(L, K)$ for $L \in \tilde{D}^{\leqslant 0}$. This follows from the triangle $\tilde{\tau}_{\geqslant 1}(K)[-1] \to \tilde{\tau}_{\leqslant 0}(K) \to K \to \tilde{\tau}_{\geqslant 1}(K)$, since $\mathrm{Hom}(L, *) = 0$ for the two extreme objects (1.16a (i)). $\mathrm{Hom}(i_*(..), K) = \mathrm{Hom}(i_*(.), j_*X) = \mathrm{Hom}(, i^!j_*X) = 0$. Thus,

$$\mathrm{Hom}(L, \tilde{\tau}_{\leqslant 0}(K)) = 0 \quad \text{if } L \in i_*(D_F^{\leqslant 0}). \tag{8.2a}$$

Since $\tilde{D}^{\leqslant 0} \subset D^{\leqslant 0}$, $i^!(\tilde{\tau}_{\leqslant 0}(K)) \in D_F^{\leqslant 0}$. By 8.2a, therefore,

$$\mathrm{Hom}(i_*i^!(\tilde{\tau}_{\leqslant 0}(K)), \tilde{\tau}_{\leqslant 0}(K)) = 0. \tag{8.2b}$$

In particular, in the triangle $i_*i^!(\tilde{\tau}_{\leqslant 0}(K)) \to \tilde{\tau}_{\leqslant 0}(K) \to j_*j^*(\tilde{\tau}_{\leqslant 0}(K))$ the first morphism is zero. The last fact actually implies that $i_*i^!(\tilde{\tau}_{\leqslant 0}(K)) = 0$: indeed, $\mathrm{Hom}(i_*i^!(\tilde{\tau}_{\leqslant 0}(K)), i_*i^!(\tilde{\tau}_{\leqslant 0}(K)) \to \mathrm{Hom}(i_*i^!(\tilde{\tau}_{\leqslant 0}(K)), \tilde{\tau}_{\leqslant 0}(K))$ is the zero map; but it also has to be an isomorphism as $\mathrm{Hom}(i_*(..), j_*(..)) = 0$.

As $i_*i^!(\tilde{\tau}_{\leqslant 0}(K)) = 0$, it follows $\tilde{\tau}_{\leqslant 0}(K) \approx j_*j^*(\tilde{\tau}_{\leqslant 0}(K))$. As $\tilde{\tau}_{\geqslant 1}(K)$ is the cone on $\tilde{\tau}_{\leqslant 0}(K) \to K$, it follows also that $j_*j^*(\tilde{\tau}_{\geqslant 1}(K)) \approx \tilde{\tau}_{\geqslant 1}(K)$. This completes the proof of 8.2.

Since j_* is a fully faithful functor 8.2 implies that we can define a t-structure $(\tilde{D}_U^{\leqslant 0}, \tilde{D}_U^{\geqslant 0})$ in $D^b(\mathrm{mod}(\Lambda_U, \Omega_U))$ by $\tilde{D}_U^{\leqslant 0} = \{X \in D^b(\mathrm{mod}(\Lambda_U, \Omega_U)) \mid j_*X \in \tilde{D}^{\leqslant 0}\}$ and $\tilde{D}_U^{\leqslant 0} = \{X \mid j_*X \in \tilde{D}^{\geqslant 0}\}$. The assumption that $\tilde{D}^{\leqslant 0} \subset D^{\leqslant 0}$ then implies $\tilde{D}_U^{\leqslant 0} \subset D_U^{\leqslant 0}$.

Let $K \in D^{\leqslant 0}$. In the triangle $i_*i^!K \to K \to j_*j^*K \to i_*i^!K[1]$ both extremes belong to $\tilde{D}^{\leqslant 0}$ (since $i_*(D_F^{\leqslant 0}) \subset \tilde{D}^{\leqslant 0}$). Thus, $K \in \tilde{D}^{\leqslant 0} \Leftrightarrow j_*j^*K \in \tilde{D}^{\leqslant 0}$. Thus, $\tilde{D}^{\leqslant 0} = \{K \in D^{\leqslant 0} \mid j_*j^*K \in \tilde{D}^{\leqslant 0}\} = \{K \in D^{\leqslant 0} \mid j^*K \in \tilde{D}_U^{\leqslant 0}\}$.

To complete the proof of Lemma 8.1, we leave the uniqueness part to the reader.

For the remaining part we assume that (Λ, Ω) is a Dynkin quiver.

Let $(\tilde{D}^{\leqslant 0}, \tilde{D}^{\geqslant 0})$ be any non-degenerate t-structure in $D^b(\mathrm{mod}(\Lambda, \Omega))$. Replacing if necessary by $(\tilde{D}^{\leqslant m}, \tilde{D}^{\geqslant m})$ for some $m \in \mathbb{Z}$, we can assume without loss that $\tilde{D}^{\leqslant 0} \subset D^{\leqslant 0}$.

(8.3) The indecomposables of $\tilde{D}^{\leqslant 0}$ are obtained by dropping a finite number, say l, of indecomposables of $D^{\leqslant 0}$.

$\mathrm{Mod}(\Lambda, \Omega)$ has only finitely many indecomposables (as (Λ, Ω) is Dynkin) and any indecomposable of $D^b(\mathrm{mod}(\Lambda, \Omega))$ is of the form $K[i]$ where $K \in \mathrm{mod}(\Lambda, \Omega)$ and $i \in \mathbb{Z}$. Since $(\tilde{D}^{\leqslant 0}, \tilde{D}^{\geqslant 0})$ is non-degenerate, $\exists n_0$ such that $m \geqslant n_0 \Rightarrow K[m] \notin \tilde{D}^{\geqslant 0}$ for any indecomposable of $\mathrm{mod}(\Lambda, \Omega)$. Thus, $\tilde{D}^{\geqslant 0} \subset D^{\geqslant -n_0}$; hence $D^{\leqslant -n_0} \subset \tilde{D}^{\leqslant 0}$. Thus the indecomposables of $D^{\leqslant 0}$ missing in $\tilde{D}^{\leqslant 0}$ are all to be found in $\mathscr{G} \cup \mathscr{G}[1] \cup \cdots \cup \mathscr{G}[n_0 - 1]$. This yields 8.3.

Let $\alpha_1, \ldots, \alpha_l, \alpha_{l+1}$ be any sequence of vertices of (Λ, Ω) such that α_1 is a sink of (Λ, Ω) and for $1 \leqslant i \leqslant l$, α_{i+1} is a sink of $(\Lambda, s_i \cdots s_1 \Omega)$. {Here, $s_j = s_{\alpha_j}$.} We now use a construction already encountered in 2.11. For $1 \leqslant \mu \leqslant l+1$, define $(\tilde{D}_{1,(\mu)}^{\leqslant 0}, \tilde{D}_{1,(\mu)}^{\geqslant 0})$ to be the image of the natural t-structure of $D^b(\mathrm{mod}(\Lambda, s_\mu \cdots s_1 \Omega))$ under the composite isomorphism $R_1^- \circ \cdots \circ R_\mu^-$ where $R_j^- : D^b(\mathrm{mod}(\Lambda, s_j \cdots s_1 \Omega)) \to D^b(\mathrm{mod}(\Lambda, s_{j-1} \cdots s_1 \Omega))$ is the isomorphism given by Proposition 1.6. Also, we set $(\tilde{D}_{1,(0)}^{\leqslant 0}, \tilde{D}_{1,(0)}^{\geqslant 0}) = (D^{\leqslant 0}, D^{\geqslant 0})$ the natural t-structure of $D^b(\mathrm{mod}(\Lambda, \Omega))$.

Choose ν_1, $1 \leqslant \nu_1 \leqslant l+1$, such that $\tilde{D}^{\leqslant 0} \subset \tilde{D}_{1,(\nu_1 - 1)}^{\leqslant 0}$ and $\tilde{D}^{\leqslant 0} \not\subset \tilde{D}_{1,(\nu_1)}^{\leqslant 0}$. Such a ν_1 exists (cf. 8.3 and 2.12). The data we are looking for (having as associated t-structure $(\tilde{D}^{\leqslant 0}, \tilde{D}^{\geqslant 0})$ shall have $\alpha_{1,\mu} = \alpha_\mu$ for $1 \leqslant \mu \leqslant \nu_1$. Apply Lemma 8.1 taking $(\Lambda, s_{\nu_1 - 1} \cdots s_1 \Omega)$ instead of (Λ, Ω) and α_{1,ν_1} instead of α. As $(\tilde{D}^{\leqslant 0}, \tilde{D}^{\geqslant 0})$ is non-degenerate it is easy to see that $(\tilde{D}_U^{\leqslant 0}, \tilde{D}_U^{\geqslant 0})$ given by Lemma 8.1 is non-degenerate. We can use induction on $\#\Lambda$ to assume that $(\tilde{D}_U^{\leqslant 0}, \tilde{D}_U^{\geqslant 0})$ is given by a data

$\{\alpha_{p,1},\ldots,\alpha_{p,v_p}|2\leqslant p\leqslant n\}$ in (Λ_U,Ω_U). Then it follows from Proposition 2.3 and Lemma 8.1 that $(\tilde{D}^{\leqslant 0},\tilde{D}^{\geqslant 0})$ is the t-structure associated to the data $\{\alpha_{p,1},\ldots,\alpha_{p,v_p}|1\leqslant p\leqslant n\}$.

Thus we have proved the following

PROPOSITION 8.4

Let (Λ,Ω) be a Dynkin quiver. Let $(D^{\leqslant 0},D^{\geqslant 0})$ be the natural t-structure in $D^b(\mathrm{mod}(\Lambda,\Omega))$ and $(\tilde{D}^{\leqslant 0},\tilde{D}^{\geqslant 0})$ any non-degenerate t-structure. Assume, as we may, that $\tilde{D}^{\leqslant 0}\subset D^{\leqslant 0}$. Then $\exists$ a data $\{\alpha_{p,1},\ldots,\alpha_{p,v_p}|1\leqslant p\leqslant n\}$ in (Λ,Ω) for which the associated data (2.9) coincides with $(\tilde{D}^{\leqslant 0},\tilde{D}^{\geqslant 0})$.

Using Theorem 7.1, we have now the following result.

Theorem 8.5. *Let (Λ,Ω) be a Dynkin quiver. Let $(\tilde{D}^{\leqslant 0},\tilde{D}^{\geqslant 0})$ be a non-degenerate t-structure in $D^b(\mathrm{mod}(\Lambda,\Omega))$, with heart $\tilde{\mathscr{G}}$. Assume real: $D^b(\tilde{\mathscr{G}})\to D^b(\mathrm{mod}(\Lambda,\Omega))$ is an equivalence. Then there exist finite dimensional k-algebras $A(1),\ldots,A(n)(n=\#\Lambda)$ such that* (i) *$A(j-1)$ is obtained by tilting $A(j)$* (ii) *$\mathrm{mod}(A_1)\approx\tilde{\mathscr{G}}$ and* (iii) *$A(n)$ is the quiver algebra of $(\Lambda,\bar{\Omega})$ (a quiver without relations) where $(\Lambda,\bar{\Omega})$ has the same underlying Dynkin graph as (Λ,Ω) but possibly with different orientations.*

In [5, § 5] similar results have been enunciated by D Happel, who proves the following: suppose (Λ,Ω) is a Dynkin quiver. Let A be an algebra such that $D^b(\mathrm{mod}\,A)\approx D^b(\mathrm{mod}\,k[\Lambda,\Omega])$. Then A is isomorphic to an iterated tilted algebra of Dynkin type. In fact one can choose a sequence of APR tilts.

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A tribute to a work of C P Ramanujam

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Abstract. In this expository article we describe some recent results on the existence and general properties of complex homology 2-cells and contractible surfaces. The motivation for these can be directly traced back to C P Ramanujam's important paper, 'A topological characterization of the affine plane as an algebraic variety'. We also discuss the related results of M Miyanishi, T Sugie and T Fujita.

Keywords. Affine surface; fundamental group; rational surface.

1. Introduction

This is an expository article that centres around a theorem of C P Ramanujam and recent developments arising out of it. In trying to solve the cancellation problem for an affine space, '$V \times \mathbb{C} \approx \mathbb{C}^3$ implies $V \approx \mathbb{C}^2$ (as an affine variety)', C P Ramanujam proved the following beautiful result in [11].

Let V be an affine, non-singular, topologically contractible surface/$\mathbb{C}$ which is simply-connected at infinity. Then V is isomorphic to $\mathbb{C}^2$ as an affine variety.

Actually, from the proof it follows that it suffices to assume only that V is affine, non-singular, the co-ordinate ring $\Gamma(V)$ is a UFD and V is simply-connected at ∞. To quote from Prof. David Mumford:

'Perhaps C P Ramanujam's most perfect piece of work in algebraic geometry is his proof that a smooth affine complex surface, which is contractible and simply-connected at ∞, is isomorphic to the plane $\mathbb{C}^2$. The proof of this is not simple and uses many techniques; in particular, it shows how well he knew his way about in the classical geometry of surfaces! What is equally astonishing is his very striking counter-example showing that the hypothesis "Simply-connected at ∞ cannot be dropped. The position of this striking example in a general theory of 4-manifolds and particularly in a general theory of the topology of algebraic surfaces is yet to be understood'. See [10].

Some recent results of A R Shastri, M Miyanishi and myself have considerably increased the understanding of smooth contractible affine surfaces. We will now describe these new results in a cronological order.

In my Ph.D. thesis I applied C P Ramanujam's method for the following question:

Let $f: \mathbb{C}^2 \to V$ be a proper morphism onto a smooth affine variety/$\mathbb{C}$. Is $V \approx \mathbb{C}^2$ as an affine variety? I was able to prove that assuming more generally that V is only normal, that V is contractible and has finite fundamental group at infinity. If further

V is smooth, then either the fundamental group at infinity is trivial or it is the binary icosahedral group of order 120. As a corollary, it follows that if degree $f \not| 120$, then $V \approx \mathbb{C}^2$. Since the method used by C P Ramanujam has been important in the proof of the above result and subsequent papers, [2], [4], [5], [12], we will describe it briefly.

One embedds an affine, normal surface $V \subset \bar{V}$ where $\bar{V}$ is a normal, projective surface which contains V as a Zariski-dense open subset such that $\bar{V}$ is smooth outside the singularities of V and $\bar{V} - V$ is a divisor with normal crossing. Let $\bar{V} - V = \bigcup_{i=1}^{r} C_i$. Here C_i are the irreducible components of $\bar{V} - V$. When V is smooth and contractible (more generally, when $H_i(V; \mathbb{Z}) = (0)$ for all $i > 0$), each $C_i \approx \mathbb{P}^1$, there is no cycle in the dual graph of $\bigcup_{i=1}^{r} C_i$ and the determinant $|C_i \cdot C_j| = \pm 1$. One can further assume (without loss of generality) that if a component C_i has self-intersection -1, then it meets at least three other components. One can construct a "nice" tubular neighbourhood T of UC_i in $\bar{V}$ (in fact, a fundamental system of such neighbourhoods) such that ∂T is a compact, C^∞ 3-manifold. In his important paper on the topology of normal 2-dimensional singularities, D Mumford gave a generators and relations presentation of $\pi_1(\partial T)$ in terms of the intersection matrix $(C_i \cdot C_j)$. It is this presentation which has proved to be extremely important in C P Ramanujam's proof (as also in the study of rational singularities,...). When $|C_i \cdot C_j| = \pm 1$, ∂T is actually a homology 3-sphere (which is the boundary of the contractible, C^∞ 4-manifold $V - \mathring{T}$). D Mumford proved the following purely group theoretical lemma:

Let G_1, G_2, G_3 be non-trivial groups and $a_i \in G_i$ be three elements. Then the group

$$\frac{G_1 * G_2 * G_2}{\langle a_1 a_2 a_3 \rangle}$$

is non-trivial.

A R Shastri and myself studied C P Ramanujam's method closely. In order to remove the restriction in the above result viz. $120 \not| \deg f$, it was necessary to generalize Mumford's group-theoretical lemma. Shastri found such a generalization which is as follows:

Let $G_1, \ldots, G_n$ be any non-trivial groups, $a_i \in G_i$ be any elements. Then (i) for $n \geqslant 4$, $G_1 * \cdots * G_n / \langle a_1 \cdot a_2 \cdots a_n \rangle$ is infnite, (ii) for $n \geqslant 3$, $G_1 * G_2 * G_3 / \langle a_1 \cdot a_2 \cdot a_3 \rangle$ is non-trivial and (iii) $G_1 * G_2 * G_2 / \langle a_1 \cdot a_2 \cdot a_3 \rangle$ is finite $\Rightarrow G_i$ are cyclic groups generated by $a_i, i = 1, 2, 3$.

Using this lemma and C P Ramanujam's method, we could prove that a smooth, contractible surface cannot have a non-trivial finite fundamental group at infinity.

Around the same time, M Miyanishi, T Sugie and T Fujita proved the cancellation theorem for $\mathbb{C}^2$. Their proof was quite different and uses the theory of logarithmic Kodaira dimension of pseudo-effective divisors. Because of the importance of this theory for their proof and the papers [1, 8, 9] and [3], we will quickly describe the statement of this decomposition.

Let X be a smooth, projective surface/$k = \bar{k}$ and D a pseudo-effective divisor on X i.e. $D \cdot H \geqslant 0$ for every ample divisor H on X. Then $\exists$ Q-divisors P and N having the properties

(i) $D \underset{\text{numerically}}{\approx} P + N$.
(ii) P is pseudo-ample i.e. $P \cdot C \geqslant 0$ for every curve C on X.
(iii) N is a non-negative Q-linear combination of irreducible curves such that the intersection form on the components of N is negative definite.
(iv) $P \cdot C = 0$ for every irreducible component of supp $\cdot N$.
(v) if D is effective, P is also effective.

Zariski had proved the existence of such a decomposition for the case when D is effective. T Fujita generalized it to the case of pseudo-effective divisors. S Iitaka has introduced the notion of logarithmic Kodaira dimension, which we now describe.

Let V be a smooth, quasi-projective surface/$\mathbb{C}$. Embed $V \subset \bar{V}$ with $\bar{V}$ a smooth, projective surface such that $\bar{V} - V$ is a divisor with normal crossings. Let D be the reduced divisor $\bar{V} - V$. Define

$$\bar{K}(V) = \max_n \dim \Phi_{|n(K_{\bar{V}} + D)|}(\bar{V}),$$

where $\Phi_{|n(K_{\bar{V}}+D)|}$ is the rational map given by the linear system $|n(K_{\bar{V}} + D)|$. Now the characterization of $\mathbb{C}^2$ due to Miyanishi, Sugie and Fujita is as follows.

Let V be a smooth, affine surface/$\mathbb{C}$ such that

(i) the co-ordinate ring $\Gamma(V)$ is a UFD.
(ii) $\Gamma(V)^* = \mathbb{C}^*$ (i.e. no non-trivial units in $\Gamma(V)$).
(iii) $\bar{K}(V) = -\infty$ i.e. $|m(K_{\bar{V}} + D)| = \phi$ for all $n \geqslant 1$. Then $V \approx \mathbb{C}^2$.

From this characterization, it is easy to see that if $f: \mathbb{C}^2 \to V$ is a proper, surjective morphism onto a smooth affine surface, then $V \approx \mathbb{C}^2$. Cancellation theorem for $\mathbb{C}^2$ also follows from the above characterization. The theory of Zariski–Fujita decomposition has proved to be very important for the classification of non-complete algebraic surfaces, as has been amply demonstrated by the work of Y Kawamata, T Fujita, M Miyanishi, S Tsunoda and others.

So the difference in C P Ramanujam's characterization of $\mathbb{C}^2$ and Miyanishi, Sugie, Fujita's characterization of $\mathbb{C}^2$ is in the hypothesis about the behaviour at infinity of V. The relationship between these two hypotheses was something of a puzzle for me. But more about it later.

M Miyanishi also proved the following nice result in [8].

Let $f: \mathbb{C}^2 \to V$ be a proper, surjective morphism onto a normal, affine surface. Then $V \approx \mathbb{C}^2/G$ where G is a "small" finite subgroup of $GL(2, \mathbb{C})$ acting linearly on $\mathbb{C}^{2'}$. His method again used the theory of logarithmic Kodaira dimension. Then A R Shastri and myself proved the same result using the topological method of C P Ramanujam (i.e. using the fundamental group at infinity). Later on Shastri and I proved the converse of the above result of Miyanishi which is at the same time a generalization of C P Ramanujam's result. We proved:

Let V be a normal, contractible surface which has finite fundamental group at infinity. Then $V \approx \mathbb{C}^2/G$ where G is a "small" finite subgroup of $GL(2, \mathbb{C})$ acting linearly on $\mathbb{C}^2$.

Because of the usefulness of the notion of fundamental group at infinity, it was desirable to have a complete classification of configurations of curves at infinity such

that the fundamental group at infinity ($\pi_1(\partial T)$ in the notation used earlier) is finite. This was done by A R Shastri in [12]. In addition to C P Ramanujam's technique, Shastri's proof uses results like Dehn's lemma and loop theorem from 3-dimensional topology! As a corollary of his classification, Shastri proved that any affine, smooth surface with finite fundamental group at infinity is rational. I also found a proof of this rationality without making use of Shastri's classification.

As mentioned earlier, the relationship between Ramanujam's method and the purely geometric method of Miyanishi, Sugie, Fujita was a mystery I wanted to solve. Finally such a connection was discovered in [3]. The result is the following:

'Let V be a smooth, affine surface with finite fundamental group at infinity. Then $\bar{K}(V) = -\infty$'.

From $\bar{K}(V) = -\infty$ again the rationality of V follows easily. Thus it appears that Ramanujam's hypothesis is stronger than that of Miyanishi, Sugie, Fujita. But the proof of the above result uses Shastri's classification mentioned above, which in turn uses C P Ramanujam's method. So Ramanujam's characterization of $\mathbb{C}^2$ does not follow from Miyanishi, Sugie, Fujita's characterization! I believe that the use of explicit classification of Shastri can be avoided in the proof of the above implication.

2. New examples of homology 2-cells

In his paper [11], C P Ramanujam found an example of a smooth, contractible surface with infinite fundamental group at infinity. The example can be described easily. In $\mathbb{CP}^2$, let C be the irreducible cubic $\{ZY^2 - X^3 = 0\}$ and D a smooth conic which intersects C in two points p, q with intersection multiplicities 5 and 1 resp. Assume p, q are not inflexion points of C. Blow-up $\mathbb{P}^2$ at q and let X be the surface obtained. Let C', D' be the proper transforms of C, D resp. Then $V = X - (C' \cup D')$ is the required example.

For a long time, this was the only non-trivial example of a smooth contractible surface. There was another reason to have more such example. This was because of a question Van de Ven asked in [13].

'Let V be a smooth, affine surface such that $H_i(V; \mathbb{Z}) = (0)$ for $i > 0$. Is V rational?'

We will call such a surface a complex homology 2-cell.

In order to see if more examples exist (and for other questions) Miyanishi and I decided to study affine surfaces systematically using the invariant $\bar{K}$. It turns out that for $\bar{K} = 0$ such surfaces do not exist. But when $\bar{K} = 1$, U Kawamata has proved that V admits a $\mathbb{C}^*$-fibration. Using the homology conditions on V, one can analyse the possible singular fibres of such a $\mathbb{C}^*$-fibration. Using all this, finally Miyanishi and I produced infinitely many non-isomorphic complex homology 2-cells and contractible surfaces. In fact, all homology 2-cells and contractible smooth surfaces V with $\bar{K}(V) = 1$ arise in the way we describe in [3]. In [3] an example of a smooth, contractible surface with $\bar{K} = 2$ is also constructed. The construction is somewhat similar to C P Ramanujam's example. This leads us to an unsolved question

Question: Are there infinitely many non-isomorphic smooth contractible surfaces V with $\bar{K}(V) = 2$?

3. Rationality of complex homology cells

Recently A R Shastri and I have proved that all smooth, complex homology 2-cells are rational. See [6, 7]. More generally, we prove the following.

Theroem. *Let X be an irreducible, smooth, projective surface/$\mathbb{C}$, with the geometric genus $P_g(X) = 0$. Let D be a reduced (not necessarily connected) curve on X which has normal crossings. Suppose each connected component of* Supp D *is simply-connected and the irreducible components of D generate the divisor class group* Pic(X). *Then X is rational.*

I will briefly describe the method. Y Miyaoka has proved an inequality between Chern-numbers for non-complete smooth surfaces (for complete case, it was also conjectured by Van de Ven). This inequality is one of the main steps in the proof of above theorem. Also the insight we gained about configurations of rational curves with unimodular determinant from C P Ramanujam's paper was very important. Detailed properties of Zariski decomposition worked out by T Fujita also play an important role in the proof. The proof is split up into two parts, to deal with surfaces of general type and elliptic type. Surprisingly, the elliptic case gave much more trouble in settling. The details of the proof are too technical to describe here.

We get the following interesting consequence of the rationality of complex homology 2-cells:

COROLLARY

For a complex homology 2-cell V, the Chow-groups of 0*-cycles and* 1*-cycles are trivial. All algebraic vector bundles on V are trivial.*

This is a generalization of the theorem of C S Seshadri on the triviality of vector bundles on $\mathbb{C}^2$. We can therefore ask the following analogue of Serre's question for contractible varieties:

Question: If V is a smooth, contractible affine variety, are all the algebraic vector bundles on V trivial?

Finally, to give a proof of the cancellation theorem for $\mathbb{C}^2$ using C P Ramanujam's method remains a cherished dream for me.

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